

Mathematics

5 Points – 1st Exam

מתמטיקה

5 יחידות לימוד – שאלון ראשון

Instructions

א. Duration of the exam: Four hours and twenty minutes.

ב. Exam structure and breakdown of points:

This exam has four parts, with a total of eight questions.

Part One: Short questions

Part Two: Induction, Sequences, and Probability

Part Three: Planar Geometry and Planar Trigonometry

Part Four: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions

Answer five questions of your choice – each question is
worth 22 points. The maximum number of points you can
earn is 100.

ג. Material that may be used during the exam:

(1) A non-graphing calculator.

Do not use the programming options of a programmable
calculator.

Use of a graphing calculator or programming options
may lead to the disqualification of your exam.

(2) Formula sheets (attached).

(3) A Hebrew–foreign language/foreign language–Hebrew
dictionary.

ד. Special instructions:

(1) Do not copy the question, but do indicate the number of
the question you are answering.

(2) Start each question on a new page.

Write the steps of the solution in the answer booklet, even
when the calculations are done with a calculator.

Explain all your work, including calculations, in detail
and in a clear and organized fashion.

Lack of detail may lower your score or may lead to the
disqualification of your exam.

הוראות

א. משך הבחינה: ארבע שעות ועשרים דקות.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה ארבעה פרקים, ובהם שמונה שאלות.

פרק ראשון: שאלות קצרות

פרק שני: אינדוקציה, סדרות והסתברות

פרק שלישי: גאומטריה וטריגונומטריה במישור

פרק רביעי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,

של פונקציות שורש, של פונקציות רציונליות

ושל פונקציות טריגונומטריות

יש לענות על חמש שאלות לבחירתכם – לכל שאלה 22 נקודות.

סך הנקודות שתוכלו לצבור לא יעלה על 100.

ג. חומר עזר מותר בשימוש:

(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון
שיש בו אפשרות תכנות.

שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול
לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי–לועזי/לועזי–עברי.

ד. הוראות מיוחדות:

(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.

(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי

הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.

יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה ברורה
ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

Write only in the answer booklet. Write "טייטה" at the top of each draft page.

If you write draft material on paper outside the answer booklet, your exam
may be disqualified.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טייטה" בראש כל עמוד המשמש

טייטה.

כתיבת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Answer five of the questions 1–8 (each question – 22 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Short Questions

1. Answer two of the four items א–ד below. If you answer more than two items, only the first two answers in your answer booklet will be graded.

א. We know that the equality $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{b}$ is satisfied for $n = 2$.
 b is a nonzero parameter.

(1) Find the value of b .

(2) Substitute $b = 4$ and prove by mathematical induction, or any other method, that the given equality is satisfied for all natural n .

ב. The figure on the right describes the graph of function $f(x)$ defined in the domain $x \leq 6$.

One of the expressions I–IV below represents function $f(x)$.

I $f(x) = x\sqrt{6-x}$

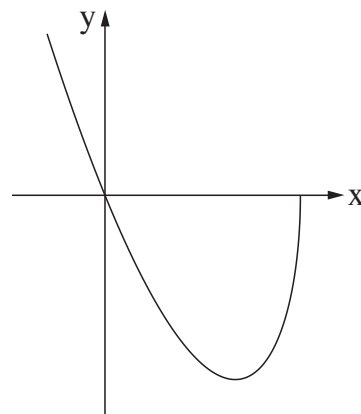
II $f(x) = -x\sqrt{6-x}$

III $f(x) = x\sqrt{x-6}$

IV $f(x) = -x\sqrt{x-6}$

(1) Determine which expression represents function $f(x)$.

Explain your answer.



Consider the function $g(x) = (f(x))^2$ defined in the domain $x \leq 6$.

(2) Calculate the area bounded by the graph of function $g(x)$ and the x-axis.

ג. The figure on the right describes the graph of function $f(x)$, defined for all x .

Function $f(x)$ has only one extremum [נקודת קיצון]

and only one horizontal asymptote whose equation is $y = 0$.

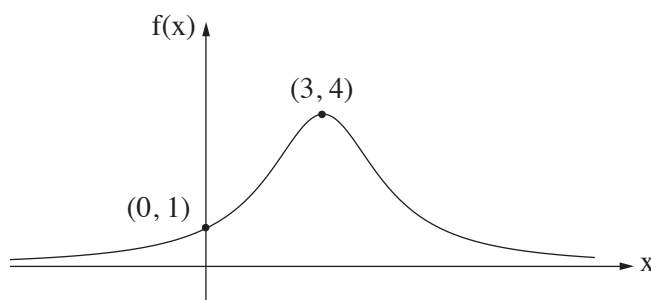
The coordinates of the extremum of function $f(x)$ and the coordinates of the intersection point of the graph of function $f(x)$ with the y-axis are shown in the figure.

Consider the function $g(x) = \frac{1}{f(x)-2}$.

(1) How many asymptotes that are perpendicular to the x-axis does function $g(x)$ have? Explain your answer.

(2) Find the equation of the asymptote of function $g(x)$ that is perpendicular to the y-axis.

(3) Sketch a graph of function $g(x)$.



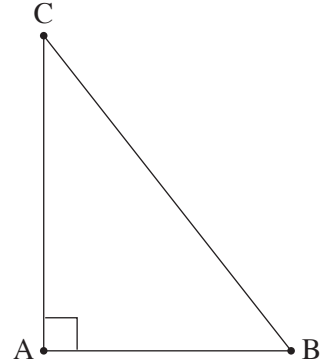
(continued on page 3)

7. The figure on the right shows a right triangle ABC ($\angle CAB = 90^\circ$).
We know that the lengths of the triangle's sides are the first three terms in a geometric sequence.

The first term in the sequence equals the length of Side AB .

We know that $AB = 1$.

- (1) Is it possible that the second term in the sequence equals the length of Side BC ? Explain your answer.
- (2) Find the length of Side BC (you may leave a root sign in your answer).



Part Two – Induction, Sequences, and Probability

2. a_n is a decreasing infinite geometric sequence of common ratio q whose terms are all positive.
 א. Consider the following statements I–III. Determine which of the statements is correct, and explain your answer.

I $-1 < q < 0$

II $0 < q < 1$

III $1 < q$

b_n is an infinite sequence whose terms satisfy $b_n = a_n + a_{n+1} + a_{n+2}$ for all natural n .

- א. Prove that Sequence b_n is geometric and that its common ratio is q .

We know that the sum of Sequence b_n is 1.96 times greater than the sum of Sequence a_n .

- א. Find the value of q .

c_n is a sequence whose terms satisfy $c_n = \frac{b_n}{4 \cdot a_n}$ for all natural n .

We know that the sum of the first k terms in the sequence is 34.79.

- א. (1) Prove that the sequence c_n is constant.
 (2) Find the value of k .

We now reverse the sign of all the terms in the even positions in Sequence c_n .

- א. Find the sum of the first k terms in Sequence c_n after the signs were reversed. Explain your answer.

3. Candidates who want to be hired by a certain company must pass three application steps in the following order: Test A, Test B, and a personal interview. Only candidates who pass Test A can take Test B, and only candidates who pass Test B get a personal interview.

Candidates who pass all three steps are hired.

The probability of a candidate passing Test A is p ($0 < p < 1$).

The probability that a candidate who passed Test A will pass Test B is 2 times greater the probability of a candidate passing Test A.

The probability that a candidate will be hired is 3 times greater than the probability that a candidate will pass both tests but not the interview.

- א. Find the probability that a candidate who passed both tests will pass the interview.

94% of the candidates are not hired.

- א. Find the value of p .
 א. We know that a certain candidate was not hired. What is the probability that this candidate passed Test A?

5 candidates are randomly selected from among all the candidates who were not hired.

- א. What is the probability that at least two of them passed Test A?

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Part Three – Planar Geometry and Planar Trigonometry

4. ABC is an equilateral triangle [משולש שווה צלעות]
inscribed in a circle.

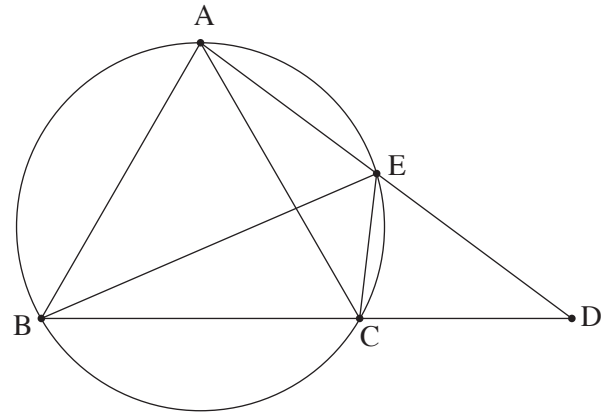
Point D is located on the extension of Side BC ,
as shown in the figure.

Segment AD intersects the circle at Point E .

א. Prove that $\angle AEB = \angle BEC = \angle CED = 60^\circ$.

ב. Prove that $\triangle AEB \sim \triangle CED$.

ג. Prove that $\frac{AE}{CE} = \frac{BC}{CD}$.



We know that the area of Triangle AEB is 2.25 times greater than the area of Triangle CED .

ד. Find by how many times the area of Triangle ABD is greater than the area of Triangle CED .

5. The figure on the right describes an isosceles triangle [משולש שווה שוקיים] ABC ($AB = AC$) .

Point E is located on Base BC such that $BE = 3 \cdot EC$.

Point D is located on Side AC such that ED is perpendicular
to AC .

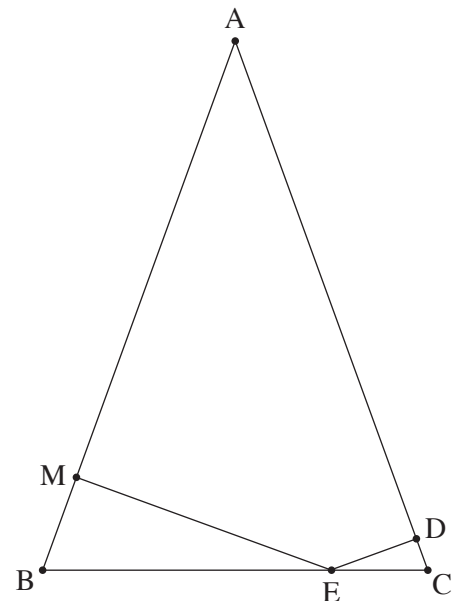
Point M is located on Side AB such that EM is
perpendicular to AB .

Let: $EC = k$, $\angle ABC = \alpha$, $30^\circ < \alpha < 90^\circ$.

א. Express the lengths of the segments DE and ME in
terms of k and α .

We know that the sum of the areas of Triangles CED and
BEM is $1.6k^2$.

ב. Find the value of α .



We know that the length of the radius of the circle circumscribing Triangle AED is 29 .

ג. Find the value of k .

Part Four – Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the function $f(x) = \frac{6(x-1)^2}{x^2 + a}$.

a is a positive parameter.

א. (1) Find the domain [תחום הגדרה] of function $f(x)$.

(2) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes (if there are any).

ב. Find the coordinates of the extrema [נקודות קיצון] of function $f(x)$, and determine what type of extrema they are. Express your answers in terms of a , if necessary.

We know that the y-coordinate of the maximum of function $f(x)$ is 7.5.

ג. Find the value of a .

Answer Items ד–ה for the value of a that you found.

ד. Sketch a graph of function $f(x)$.

$f'(x)$ is the derivative function of function $f(x)$.

Function $g(x)$ satisfies $g(x) = \frac{-f'(x)}{(f(x))^2}$.

ה. (1) Find the domain of function $g(x)$.

(2) Find the positive and negative intervals of function $g(x)$.

(3) Calculate the area bounded by the graph of function $g(x)$ and the x-axis on the interval $-a \leq x \leq 0$.

7. Consider the function $f(x) = (\sin x)^2 \cdot \cos(2x)$, defined on the domain $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

א. Prove that function $f(x)$ is even.

ב. Find the coordinates of all the extrema [נקודות קיצון] of function $f(x)$, and determine what type of extrema they are.

ג. Sketch a graph of function $f(x)$.

The derivative function $f'(x)$ is defined on the domain $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

Consider the function $g(x) = \sqrt{a \cdot f'(x)}$.

a is a negative parameter.

ד. (1) Find the domain [תחום הגדרה] of function $g(x)$.

(2) Find the coordinates of the intersection points of the graph of function $g(x)$ with the x-axis.

We know that function $f(x)$ has only one inflection point [נקודת פיתול] in the domain of function $g(x)$.

ה. Sketch a graph of function $g(x)$.

8. The figure on the right describes the graph of function

$$f(x) = (4x - 2)^3 \text{ and the line } y = 3x + b.$$

b is a positive parameter.

Point A is located on the graph of function $f(x)$ such that its x -coordinate is smaller than the x -coordinate of the intersection point of the graph of function $f(x)$ with the given line.

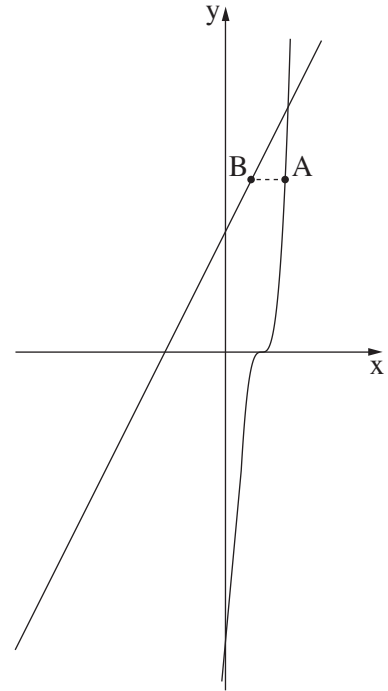
Point B is located on the given line such that Segment AB is parallel to the x -axis.

Let t be the x -coordinate of Point A .

We know that $0 \leq t \leq 0.7$.

א. Express the x -coordinate of Point B in terms of b and t .

- ב. (1) Find the value of t for which the length of Segment AB is minimum.
(2) Find the value of t for which the length of Segment AB is maximum.



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך