

Mathematics

5 Points – 1st Exam

מתמטיקה

5 יחידות לימוד – שאלון ראשון

Instructions

הוראות

א. Duration of the exam: Three and three quarter hours.

א. משך הבחינה: שלוש שעות וארבעים וחמש דקות.

ב. Exam structure and breakdown of points:

ב. מבנה השאלון ומפתח ההערכה:

This exam has four parts and a total of eight questions.

בשאלון זה ארבעה פרקים, ובהם שמונה שאלות.

Part One: Short questions

פרק ראשון: שאלות קצרות

Part Two: Induction, Sequences, and Probability

פרק שני: אינדוקציה, סדרות והסתברות

Part Three: Planar Geometry and Planar Trigonometry

פרק שלישי: גאומטריה וטריגונומטריה במישור

Part Four: Differential and Integral Calculus of

פרק רביעי: חשבון דיפרנציאלי ואינטגרלי של פולינומים, של

Polynomials, Root Functions, Rational

פונקציות שורש, של פונקציות רציונליות ושל

Functions, and Trigonometric Functions

פונקציות טריגונומטריות

Answer five questions, at least one question from Part One

יש לענות על חמש שאלות, על שאלה אחת לפחות מן הפרק הראשון או

or Part Two, and at least one question from each of

השני ועל שאלה אחת לפחות מכל אחד מן הפרקים השלישי והרביעי.

Part Three and Part Four.

$5 \times 20 = 100$ נקודות.

$5 \times 20 = 100$ points.

ג. Material that may be used during the exam:

ג. חומר עזר מותר בשימוש:

(1) A non-graphing calculator.

(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון

Do not use the programming options of a programmable calculator.

שיש בו אפשרות תכנות.

Use of a graphing calculator or programming options may lead to the disqualification of your exam.

שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.

(2) Formula sheets (attached).

(2) דפי נוסחאות (מצורפים).

(3) A Hebrew–foreign language/foreign language–Hebrew dictionary.

(3) מילון עברי–לועזי/לועזי–עברי.

ד. Special instructions:

ד. הוראות מיוחדות:

(1) Do not copy the question, but do indicate the number of the question you are answering.

(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.

(2) Start each question on a new page.

(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי

Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator.

הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.

Explain all your work, including calculations, in detail and in a clear and organized fashion.

יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה ברורה ומסודרת.

Lack of detail may lower your score or may lead to the disqualification of your exam.

חסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

Write only in the answer booklet. Write "טיוטה" at the top of each draft page.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טיוטה" בראש כל עמוד המשמש

If you write draft material on paper outside the answer booklet, your exam

טיוטה.

may be disqualified.

כתיבת טיוטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Answer five of the questions 1–8, at least one question from Part One or Part Two, and at least one question from each of Part Three and Part Four (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Short Questions

1. Answer two of the four items א–ג below. If you answer more than two items, only the first two answers in your answer booklet will be graded.

א. Prove by mathematical induction, or by any other method, that the following equality is satisfied for all natural n :

$$\frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$$

ב. Jonathan has two types of books in the bookcase in his room: comic books and nonfiction books.

60% of his books are comic books and the rest are nonfiction books.

70% of his books are in Hebrew and the rest are in English.

Consider the two statements I and II. Determine whether each is true or false, and explain your answers.

I Jonathan has no Hebrew comic books.

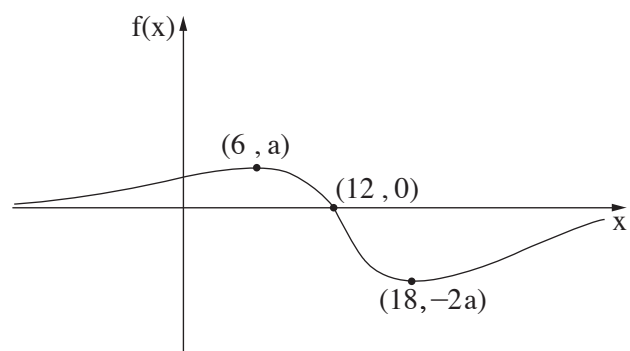
II The probability of randomly selecting a Hebrew book from among the comic books is greater than the probability of randomly selecting a comic book from among the Hebrew books.

ג. The figure on the right describes the graph of function $f(x)$, defined for all x .

Function $f(x)$ has only one asymptote, whose equation is $y = 0$, and only one intersection point with the x -axis, whose coordinates are $(12, 0)$.

The coordinates of all the extrema [נקודות קיצון] of function $f(x)$ are shown in the figure.

a is a parameter greater than 1.



Consider function $g(x) = \frac{1}{(f(x))^2}$.

(1) Find the domain [תחום הגדרה] of function $g(x)$.

(2) Sketch a graph of function $g(x)$.

We know that the line $y = \frac{a}{27}$ and the graph of function $g(x)$ have exactly 3 common points.

(3) Find the value of a .

(continued on page 3)

7. Consider the function $f(x) = \frac{-30x^2}{(x^3 - 8)^2}$ shown in the figure on the right, defined on the domain $x \neq 2$.

The figure shows all the extrema [נקודות קיצון] of the graph of function $f(x)$, and all the intersection points of the graph of function $f(x)$ with the x -axis.

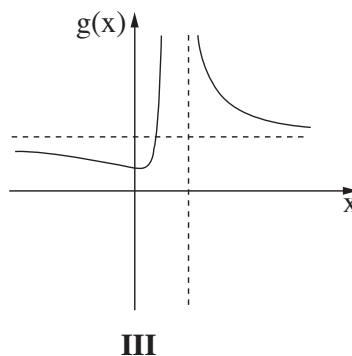
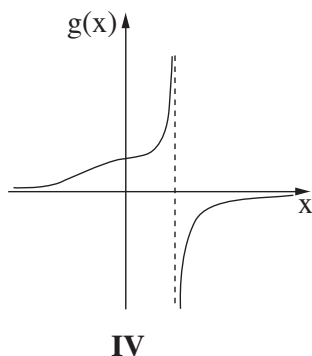
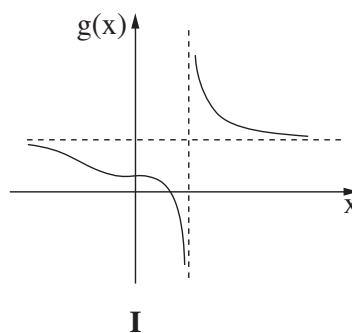
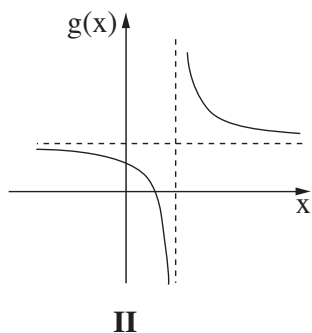
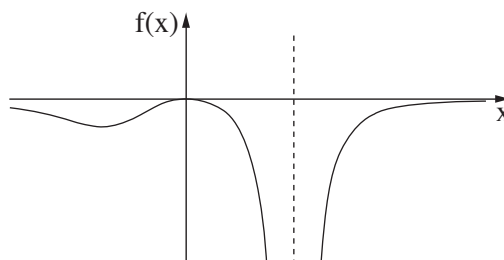
$g(x)$ is a function whose derivative satisfies

$$g'(x) = f(x).$$

Function $g(x)$ is defined on the domain $x \neq 2$.

We know that the graph of function $g(x)$ passes through the point $(0, 0.75)$.

- (1) Find a function $g(x)$ that satisfies these conditions.
- (2) One of the graphs I–IV below describes a function $g(x)$ that satisfies these conditions. Determine which one it is, and explain your answer.



Part Two – Induction, Sequences, and Probability

2. a_n is an increasing infinite geometric sequence of common ratio q .
 b_n is an increasing infinite geometric sequence of common ratio $2q$.
 c_n is an infinite sequence whose terms satisfy $c_n = a_n \cdot b_n$ for all natural n .
 א. Prove that c_n is a geometric sequence, and express its common ratio in terms of q .

We know that $c_2 = \frac{1}{8} \cdot (a_1)^2$, $a_1 = b_1$.

- א. Find the value of q .
 א. (1) Is the value of a_1 positive or negative? Explain your answer.
 (2) Is Sequence c_n increasing or decreasing? Explain your answer.

Let S_1 be the sum of Sequence a_n , let S_2 be the sum of Sequence b_n , and let S_3 be the sum of Sequence c_n .

We know that $S_1 + S_2 + S_3 = 434$.

- א. Find the value of a_1 .

3. Bag A and Bag B contain two types of balls: soft balls and hard balls.
 Bag A contains 10 balls, of which 4 balls are soft and the rest are hard.
 Bag B contains 12 balls, of which x balls are soft and the rest are hard.
 Galit randomly selects a bag and takes out one random ball. Next, she returns the ball to the bag and randomly takes out a second ball from the same bag (removal with replacement).
 We know that the probability that Galit took out two soft balls is $\frac{41}{200}$.
 א. Find the value of x .
 א. We know Galit took out two balls of the same type. What is the probability that she took out two balls from Bag A ?

Galit performs the following procedure 4 times:

She randomly selects a bag and takes out one random ball. Next, she returns the ball to the bag and randomly takes out a second ball from the same bag (removal with replacement).

- א. What is the probability that Galit took out only hard balls exactly two of the times?
 א. What is the probability that Galit took out only hard balls two of the times and only soft balls two of the times?

Part Three – Planar Geometry and Planar Trigonometry

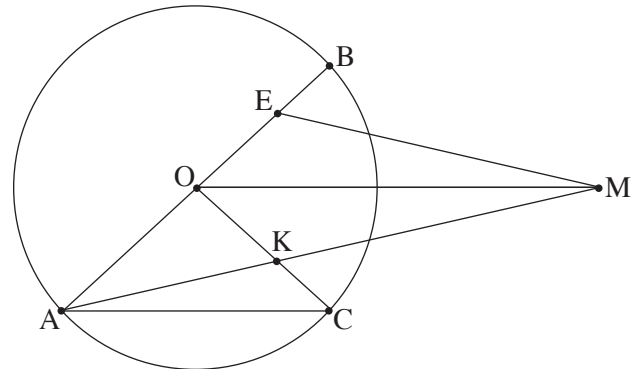
4. The figure on the right describes a circle of center O .

AB is a diameter in the circle.

Point C is located on the circle.

Point M is located outside the circle, such that Segment AM intersects Segment CO at Point K .

Point E is located on Segment BO , such that Quadrilateral $EMKO$ is a kite ($OK = OE$, $MK = ME$).



א. Prove that $BC \parallel EK$.

ב. Prove that $OM \parallel AC$.

We know that $\frac{BC}{EK} = \frac{5}{3}$.

ג. Find the value of $\frac{OM}{AC}$.

Let S be the area of Kite $EMKO$.

ד. Express the area of Triangle AOC in terms of S .

5. The figure on the right describes a trapezoid $ABCD$ ($AB \parallel DC$).

We know that the length of Base AB is equal to the length of Side BC .

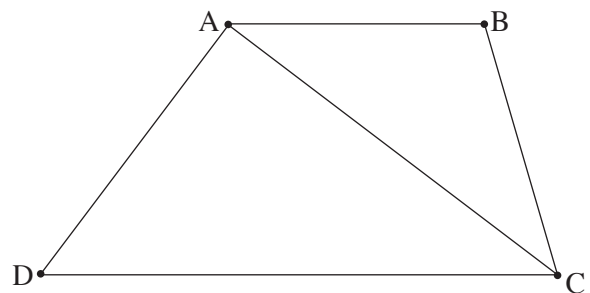
Let $AB = BC = k$, $\angle ACB = \alpha$, $0^\circ < \alpha < 90^\circ$.

א. Show that $AC = 2k \cdot \cos \alpha$.

We know that $AD = 1.2k$, $DC = 2k$.

ב. Find the value of α .

ג. Find the size of Angle ADC .



The extensions of Sides DA and CB intersect at Point E .

We know that the length of the radius of the circle inscribed in Triangle EDC is 14.

ד. Find the value of k .

Part Four – Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the function $f(x) = \frac{ax}{(x-3)^2}$, defined on the domain $x \neq 3$.
 a is a positive parameter.
- א. Answer Items (1)–(2). Express your answers in terms of a , if necessary.
- (1) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes.
- (2) Find the coordinates of the extremum [נקודת קיצון] of function $f(x)$, and determine what type of extremum it is.
- ב. Sketch a graph of function $f(x)$.

Consider the function $g(x) = f(x) + 0.5$, defined on the domain $x \neq 3$.

We know that the graph of function $g(x)$ has exactly one common point with the x -axis.

- ג. Find the value of a .

Substitute in function $g(x)$ the value of a that you found, and answer Item τ .

Consider the function $h(x) = g(x) \cdot g'(x)$, defined on the domain $x \neq 3$.

- ד. (1) Give the positive and negative intervals of function $h(x)$.
- (2) Calculate the area bounded by the graph of function $h(x)$ and the x -axis on the interval $-9 \leq x \leq 0$.

7. Consider the function $f(x) = (1 + \cos x) \cdot (-1 + b \cos x)$, defined on the domain $-\pi \leq x \leq \pi$.
 b is a parameter.

We know that $0 < b < 1$.

- א. Is function $f(x)$ even or odd? Explain your answer.
- ב. Find the coordinates of the intersection points of the graph of function $f(x)$ with the x -axis.

We know that function $f(x)$ has an extremum [נקודת קיצון] at the point where $x = \frac{\pi}{3}$.

- ג. Find the value of b .

Substitute $b = \frac{1}{2}$ in function $f(x)$ and answer Items τ – η .

- ד. (1) Find the coordinates of the extrema [נקודות קיצון] of function $f(x)$, and determine what type of extrema they are.
- (2) Sketch a graph of function $f(x)$.

Consider the function $h(x) = f(x) + |f(x)|$, defined on the domain $-\pi \leq x \leq \pi$.

Point A is any point located on the graph of function $h(x)$.

- ה. Is the y -coordinate of Point A positive, negative, equal to 0, or impossible to determine?
Explain your answer.

(continued on page 7)

8. The figure on the right describes the graph of function $f(x) = 2\sqrt{x^2 - 1}$.

Consider the given line $y = 3x - 3$.

- א. Find the domain [תחום הגדרה] of function $f(x)$.
- ב. Find the coordinates of the two intersection points of the graph of function $f(x)$ with the given line.

Point A is located on the graph of function $f(x)$ on the interval between the two points you found in Item ב.

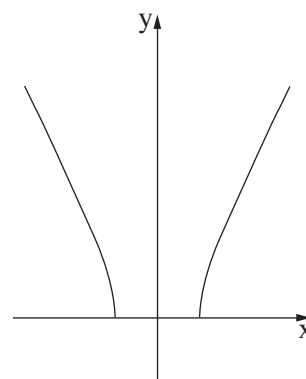
Two lines are drawn through Point A :

A line parallel to the y-axis that intersects the given line at Point B .

A line parallel to the x-axis that intersects the given line at Point C .

Let t be the x-coordinate of Point A .

- א. (1) Express the length of Segment AB in terms of t .
- (2) Express the length of Segment AC in terms of t .
- ב. Find the value of t for which the sum of the lengths of Segments AB and AC is greatest (maximum).



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך