

תוכנית חדשה

Mathematics

5 Points – 1st Exam

מתמטיקה

5 יחידות לימוד - שאלון ראשון

Instructions

א. Duration of the exam: Three hours and forty-five minutes.

ב. Exam structure and breakdown of points:

This exam has four parts with a total of eight questions.

Part One: Short questions

Part Two: Induction, Sequences, and Probability

Part Three: Planar Geometry and Planar Trigonometry

Part Four: Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

Answer five questions, at least one question from Part One or Part Two, and at least one question from each of Part Three and Part Four.

$5 \times 20 = 100$ points.

ג. Material that may be used during the exam:

(1) A non-graphing calculator. Do not use the programming options of a programmable calculator.

Use of a graphing calculator or programming options may lead to the disqualification of your exam.

(2) Formula sheets (attached).

(3) A Hebrew–foreign language/foreign language–Hebrew dictionary.

ד. Special instructions:

(1) Do not copy the question, but do indicate the number of the question you are answering.

(2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator.

Explain all your work, including calculations, in detail and in a clear and organized fashion.

Lack of detail may lower your score or may lead to the disqualification of your exam.

Write only in the answer booklet. Write "טייטה" at the top of each draft page.

If you write draft material on paper outside the answer booklet, your exam may be disqualified.

א. משך הבחינה: שלוש שעות וארבעים וחמש דקות.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה ארבעה פרקים, ובהם שמונה שאלות.

פרק ראשון – שאלות קצרות

פרק שני – אינדוקציה, סדרות והסתברות

פרק שלישי – גאומטריה וטריגונומטריה במישור

פרק רביעי – חשבון דיפרנציאלי ואינטגרלי של פולינומים, של פונקציות שורש, של פונקציות רציונליות ושל פונקציות טריגונומטריות

יש לענות על חמש שאלות, על שאלה אחת לפחות מן הפרק הראשון או השני ועל שאלה אחת לפחות מכל אחד מן הפרקים השלישי והרביעי. $20 \times 5 = 100$ נקודות.

ג. חומר עזר מותר בשימוש:

(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון שיש בו אפשרות תכנות.

שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי–לועזי/לועזי–עברי.

ד. הוראות מיוחדות:

(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.

(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.

יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה ברורה ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טייטה" בראש כל עמוד

המשמש טייטה.

כתובת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Answer five of the questions 1–8, at least one question from Part One or Part Two, and at least one question from each of Part Three and Part Four (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Short Questions

1. Answer two of the four items א–ג below. If you answer more than two items, only the first two answers in your answer booklet will be graded.

א. The figure to the right describes a circle of center O and radius R .

Two tangents that are perpendicular to each other are drawn through Points A and E on the circle.

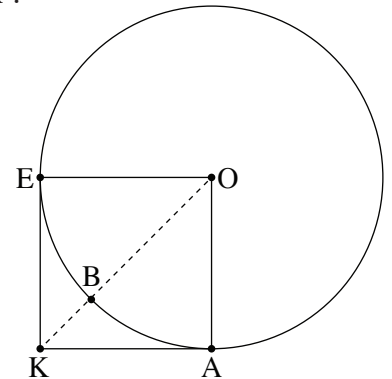
The tangents intersect at Point K .

Segment KO intersects the circle at Point B .

(1) Find how many times larger the area of Triangle AOB is than the area of Triangle AKB .

We know that the area of Triangle AOB is 16.

(2) Find the length of the height to Side AK in Triangle AKB .



ב. In a chess competition, two players – Yair and Nadav – compete against each other.

Each game can end in one of 3 ways: Yair wins, Nadav wins, or a draw.

In each game, the probability that Yair will win is constant and 2 times greater than the probability that Nadav will win.

Let p denote the probability that Nadav will win one game.

Yair and Nadav play two games.

(1) Express in terms of p the probability that one of the players will achieve two wins.

We know that if neither player achieved two wins, the probability that each of the games will end in a draw is $\frac{9}{31}$.

(2) Determine which of the options I–III below is the value of p . Explain your answer.

I. $\frac{1}{10}$

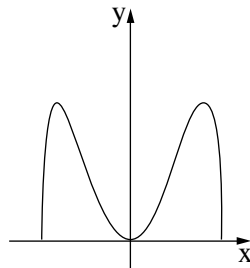
II. $\frac{11}{27}$

III. $\frac{1}{6}$

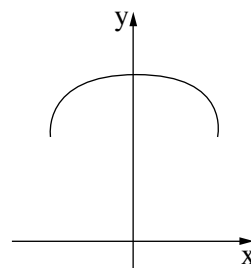
ג. Consider the function $f(x) = \sqrt{x^2 - 1} + 2\sqrt{16 - x^2}$.

(1) Find the domain [תחום ההגדרה] of function $f(x)$.

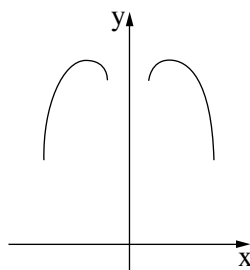
(2) One of the graphs I–IV below describes function $f(x)$. Determine which one it is and explain your answer.



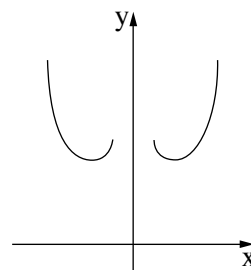
II



I



IV



III

ד. Consider an infinite sequence whose terms satisfy $a_n = \frac{\cos(\pi \cdot 2^n)}{5 \cdot 2^n}$ for every natural number n .

(1) Find the first three terms in Sequence a_n .

(2) Prove that Sequence a_n is a geometric sequence, and find its common ratio.

(3) Find the sum of Sequence a_n .

Part Two – Induction, Sequences, and Probability

2. a_n is a decreasing infinite geometric sequence whose sum converges.

The common ratio of this sequence is q .

Let T denote the sum of the odd-positioned terms in this sequence,

and let R denote the sum of the even-positioned terms in this sequence.

We know that $\frac{T^2}{R^2} = 16$.

- א. Find the value of q .

Consider Sequence b_n whose terms satisfy $b_n = \frac{1+q}{a_n \cdot a_{n+1}}$ for every natural number n .

We know that $b_1 = 1$.

- ב. Find the value of a_1 .

- ג. Express b_n in terms of n .

- ד. Prove by induction or any other method that the following equality is satisfied for every natural number n :

$$b_1 + b_2 + \dots + b_n = \frac{16^n - 1}{15}.$$

3. In the eleventh grade of a large urban high school, some of the students are counselors in a youth movement and the rest are not counselors.

In this grade, some of the students have a driver's license and the rest do not have a driver's license.

Half of the students who have a driver's license are counselors in a youth movement.

The percentage of counselors among students who have a license is equal to the percentage of students who do not have a license among counselors.

A student is randomly selected from among the students in this grade.

The probability that the selected student is a counselor or has a license is 0.45.

- א. Find what percentage of students in this grade are counselors.

- ב. A student is randomly selected from among the students who do not have a license.

What is the probability that this student is a counselor?

5 students are randomly selected from among the students who do not have a license.

- ג. (1) What is the probability that more than one counselor was selected?

- (2) We know that more than one counselor was selected. What is the probability that the number of counselors selected is even?

Part Three – Planar Geometry and Planar Trigonometry

4. The figure to the right describes an obtuse triangle ABC inscribed in a circle.

Point K is located on the extension of Side AB such that $AB = BK$.

The extension of Segment KC intersects the circle at an additional point, Point F .

We know that $\angle BAC = \angle BCA$.

- א. Prove that AF is a diameter of the circle.

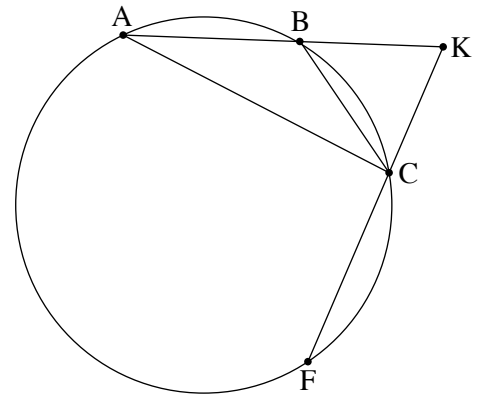
The chords BF and AC intersect at Point D .

- ב. Prove that Triangle ADK is an isosceles triangle.

- ג. (1) Prove that the quadrilateral $BDCK$ can be inscribed in a circle.

(2) Prove that $\angle DKC = \angle FAC$.

- ד. Prove that $AC \cdot AD = KC \cdot AF$.



5. The figure to the right describes Kite $ABCD$ ($AB = AD$, $CB = CD$).

We know that $\angle BAD = \angle CDB = 2\alpha$, $0^\circ < \alpha < 45^\circ$.

Let $AB = k$.

- א. Express the radius of the circle circumscribing Triangle CDB in terms of k and α .

We know that the radius of the circle circumscribing Triangle ABD is $\frac{4}{3}$ times the radius of the circle circumscribing Triangle CDB .

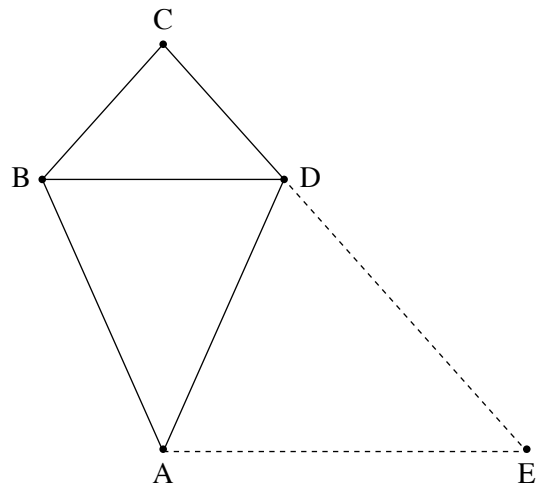
- ב. Find the size of Angle BAD .

Point O is the center of the circle circumscribing Triangle ABD .

A line is drawn through Point A , parallel to BD , and intersecting the extension of Side CD at Point E .

We know that the area of Triangle AOE is 70.

- ג. Find the value of k .



Part Four – Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the function $f(x) = \frac{\sqrt{5x^2 - kx}}{x^3}$, k is a positive parameter.

א. Express the domain [תחום ההגדרה] of function $f(x)$ in terms of k .

Consider the function $h(x) = -f(x)$.

The x -coordinate of the intersection point of the graph of function $f(x)$ with the graph of function $h(x)$ is 1.6.

ב. Find the value of k .

Substitute $k = 8$ in function $f(x)$, and answer Items ג–ו.

- ג. (1) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes.
- (2) Find the coordinates of all the extrema [נקודות קיצון] of function $f(x)$, and determine what type of extrema they are.
- (3) Sketch a graph of function $f(x)$.

Consider function $g(x) = (f(x))^2$ whose domain is the same as the domain of function $f(x)$.

ד. Sketch a graph of function $g(x)$.

a is a parameter greater than 2.

- ה. Is the value of the expression $\int_a^{a+1} g(x) dx$ equal to, greater than, or less than the value of the expression $\int_a^{a+1} f(x) dx$? Explain your answer.
- ו. Calculate the area bounded by the graph of function $g(x)$, the x -axis, and the lines $x = 2$ and $x = 6$.

7. Consider the function $f(x) = (\sin x)^2 + \cos x - 1$, defined on the domain $-\pi \leq x \leq \pi$.

א. Prove that function $f(x)$ is even.

- ב. (1) Find the coordinates of the intersection points of the graph of function $f(x)$ with the x -axis.
- (2) Find the coordinates of all the extrema [נקודות קיצון] of function $f(x)$ and determine what type of extrema they are.
- ג. Sketch a graph of function $f(x)$.

Consider the function $g(x) = \frac{1}{f(x) + b}$, b is a parameter.

We know that function $g(x)$ has exactly two vertical asymptotes.

ד. Write two possible values for b , one positive and the other negative.

Substitute the negative value you found for b in function $g(x)$, and answer Items ה–ו.

- ה. (1) Find the domain [תחום ההגדרה] of function $g(x)$.
- (2) Find the coordinates of all the extrema of function $g(x)$, and determine what type of extrema they are.
- ו. Sketch a graph of function $g(x)$.

8. Consider the function $f(x) = kx^3 - 3x^2 + 8kx$, defined for all x .

k is a non-zero parameter.

Point A is the inflection point [נקודת הפיתול] of function $f(x)$.

א. Express the coordinates of Point A in terms of k .

We know that Point A is located in the first quadrant.

ב. Find the range of possible values of k .

Through Point A , a perpendicular is dropped to the x -axis, intersecting it at Point B , and a perpendicular is dropped to the y -axis intersecting it at Point C .

Consider Square I whose side length is equal to Segment AB , and Square II whose side length is equal to Segment AC .

ג. Find the value of k for which the sum of the areas of Squares I and II is minimum (smallest). You can leave a square root in your answer.

Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך