

Note: This exam has special instructions.
Answer the questions according to these instructions.

שימו לב: בבחינה זו יש הנחיות מיוחדות.
יש לענות על השאלות על פי הנחיות אלה.

Mathematics

5 Points – 1st Exam

מתמטיקה

5 יחידות לימוד – שאלון ראשון

Instructions

הוראות

א. Duration of the exam: Four and a quarter hours.

א. משך הבחינה: ארבע שעות ורבע.

ב. Exam structure and breakdown of points:

ב. מבנה השאלון ומפתח ההערכה:

This exam has four parts and a total of eight questions.

בשאלון זה ארבעה פרקים, ובהם שמונה שאלות.

Part One: Short questions

פרק ראשון – שאלות קצרות

Part Two: Induction, Sequences, and Probability

פרק שני – אינדוקציה, סדרות והסתברות

Part Three: Planar Geometry and Planar Trigonometry

פרק שלישי – גאומטריה וטריגונומטריה במישור

Part Four: Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

פרק רביעי – חשבון דיפרנציאלי ואינטגרלי של פולינומים, של פונקציות שורש, של פונקציות רציונליות ושל פונקציות טריגונומטריות

Answer five questions, one question at least from Part One or Part Two, and at least one question from each of Part Three and Part Four.

יש לענות על חמש שאלות, על שאלה אחת לפחות מן הפרק הראשון או השני ועל שאלה אחת לפחות מכל אחד מן הפרקים השלישי והרביעי.

$5 \times 20 = 100$ points.

$20 \times 5 = 100$ נקודות.

ג. Material that may be used during the exam:

ג. חומר עזר מותר בשימוש:

(1) A non-graphing calculator.

(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון

Do not use the programming options of a programmable calculator.

שיש בו אפשרות תכנות.

Use of a graphing calculator or programming options may lead to the disqualification of your exam.

שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.

(2) Formula sheets (attached).

(2) דפי נוסחאות (מצורפים).

(3) A Hebrew–foreign language/foreign language–Hebrew dictionary.

(3) מילון עברי–לועזי/לועזי–עברי.

ד. Special instructions:

ד. הוראות מיוחדות:

(1) Do not copy the question, but do indicate the number of the question you are answering.

(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.

(2) Start each question on a new page.

(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי

Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator.

הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.

Explain all your work, including calculations, in detail and in a clear and organized fashion.

יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה ברורה ומסודרת.

Lack of detail may lower your score or may lead to the disqualification of your exam.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

Write only in the answer booklet. Write "טיוטה" at the top of each draft page.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טיוטה" בראש כל עמוד המשמש

If you write draft material on paper outside the answer booklet, your exam may be disqualified.

טיוטה.

כתיבת טיוטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Answer five of the questions 1–8, at least one question from Part One or Part Two, and at least one question from each of Part Three and Part Four (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Short Questions

1. Answer two of the four items א–ג below. If you answer more than two items, only the first two answers in your answer booklet will be graded.

א. Prove by mathematical induction, or any other method, that the following expression is true for all natural n :

$$\frac{2}{3} + \frac{6}{9} + \frac{10}{27} + \dots + \frac{4n-2}{3^n} = 2 - \frac{2n+2}{3^n}$$

ב. Consider the trapezoid ABCD ($BC \parallel AD$) and the trapezoid BCDE ($CD \parallel BE$) in the figure on the right.

Point M is the intersection point of the sides AD and BE.

We know that $EM = BM$, $ED \perp AD$, $AB \perp BM$;

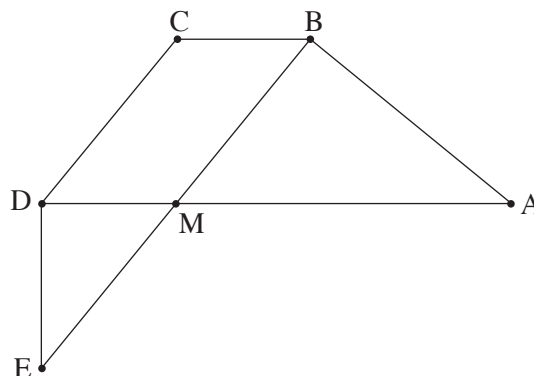
Segment AM is 2.5 times longer than Segment DM.

Let $\angle EMD = \alpha$.

(1) Find the value of α .

We know that $DM = 4$.

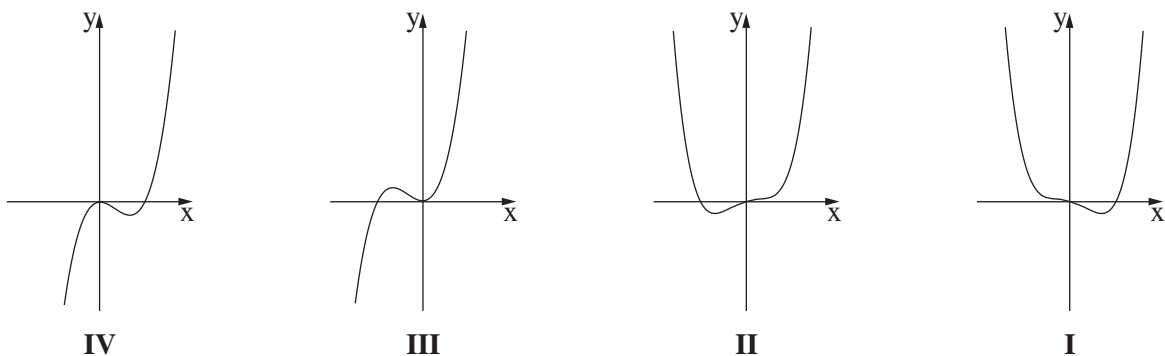
(2) Calculate the area of the trapezoid ABCD.



- ג. Consider the functions $f(x) = x^n(x-1)$ and $g(x) = x^n(x+1)$, defined for all x .

The parameter n is a natural number greater than 1.

Each of the graphs I–IV below represents one of the functions $f(x)$ or $g(x)$, where n is either even or odd.



- (1) Match each of the graphs I–IV with the correct function, either $f(x)$ or $g(x)$, where n is either even or odd.

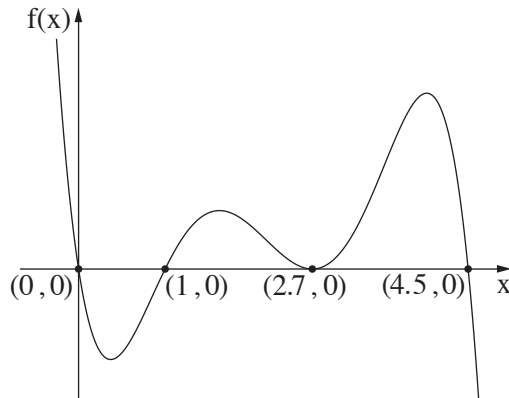
- (2) Substitute $n = 4$ and calculate the area bounded by the graph of function $g(x)$ and the x-axis.

- ז. The figure on the right describes the function $f(x)$ defined for all x .

The figure shows all the coordinates of the intersection points of the graph of function $f(x)$ with the x-axis.

Function $g(x)$ is defined on the interval $x \geq 0$ and

satisfies $g(x) = \int_0^x f(t) dt$.



- (1) Find the x-coordinates of the extrema [נקודות קיצון] of function $g(x)$, and determine what type of extrema they are.

Let S be the area bounded by the graph of function $f(x)$ and the x-axis on the interval $1 \leq x \leq 2.7$.

We know that $g(2.7) = 0$, $g(4.5) = 3 \cdot S$.

- (2) Express in terms of S the area bounded by the graph of function $f(x)$ and the x-axis on the interval $0 \leq x \leq 4.5$.

Part Two – Induction, Sequences, and Probability

2. a_n is an increasing infinite geometric sequence of common ratio q .

$$0 < q < 1.$$

- א. Determine whether all the terms of Sequence a_n are negative or positive. Explain your answer.

b_n is an infinite geometric sequence whose common ratio is q , too.

c_n is an infinite sequence which satisfies $c_n = 2b_n - a_n$ ($c_n \neq 0$) for all natural n .

- ב. Prove that Sequence c_n is a geometric sequence, and express its common ratio in terms of q .

We know that $c_1 = 5\frac{2}{3}$, and that the sum of the terms of Sequence c_n is 17 times greater than the sum of the terms of Sequence b_n .

- ג. Find the value of a_1 .

We know that the sum of the terms in the even positions in Sequence a_n is greater by 4 than the sum of the terms in the odd positions in Sequence a_n .

- ד. Find the value of q .

3. A certain large packing house packs only two types of fruit: oranges and lemons.

Some of these fruit are intended for export and the rest are not.

The probability of randomly selecting two oranges from all the fruit in the packing house is 0.4096.

We randomly select one fruit from among all the fruit in the packing house.

- א. What is the probability of selecting a lemon?

The probability of randomly selecting an orange from among the fruit intended for export is $\frac{3}{5}$.

The probability of randomly selecting a lemon from among the fruit not intended for export is $\frac{4}{15}$.

We randomly select one fruit from among all the fruit in the packing house.

- ב. What is the probability of selecting a fruit intended for export?

We randomly select two fruits from among all the fruit in the packing house.

- ג. We know that the two selected fruits are lemons. What is the probability that one of them is intended for export and the other is not?

We randomly select 4 fruits from among all the fruit in the packing house. We know that at least one of them is intended for export.

- ד. What is the probability that, at most, 3 of the selected fruits are intended for export?

Part Three – Planar Geometry and Planar Trigonometry

4. AB is a diameter of a circle of center O .

Chord CK intersects the radius AO at Point E (see figure).

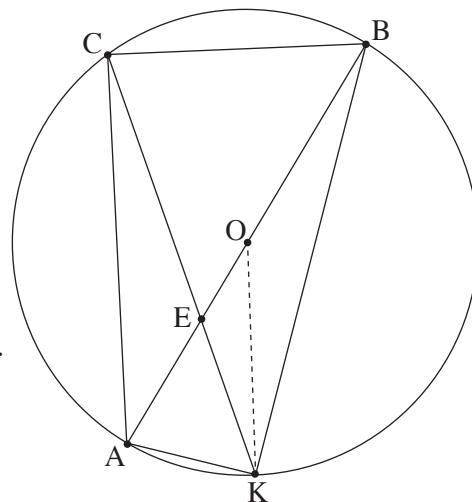
We know that $\angle EKO = \angle ABK$.

- א. Prove that $\triangle ACE \sim \triangle OKE$.

The extension of Segment KO intersects Chord CB at Point P .

We know that $PO = 4$, and that the radius of the circle is 4.8 .

- ב. (1) Prove that PO is a midsegment [קטע אמצעים] in Triangle ABC .
(2) Find the length of Segment EO .
ג. Find the ratio between the area of Triangle ACE and the area of Triangle AOK .



5. Consider the right triangle ABC ($\angle ABC = 90^\circ$) in the figure on the right.

A second triangle ACD is constructed on Side AC such that AC bisects Angle BAD .

Let $AB = k$, $\angle CAB = \alpha$.

- א. Express the length of AC in terms of k and α .

We know that $AD = 1.5 \cdot k$, $CD = \sqrt{3.25} \cdot k$.

- ב. Find the value of α .

Substitute $\alpha = 60^\circ$ and answer Items ג–ד.

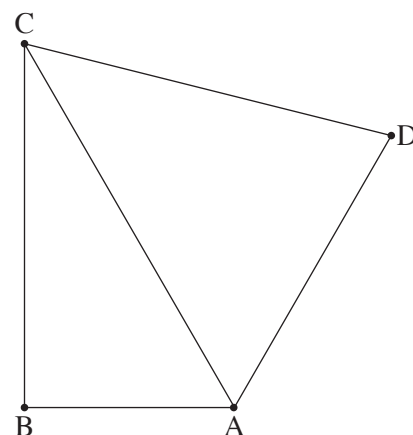
Point M is the center of the circle inscribed in Triangle ABC .

- ג. Express the radius of the circle inscribed in Triangle ABC in terms of k .

Point E is the center of the circle circumscribing Triangle ABC .

We know that $ME = 6$.

- ד. Calculate the value of k .



Part Four – Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the function $f(x) = \frac{2ax}{(x^2 - 16)^2}$; a is a positive parameter.

- א. (1) Find the domain [תחום ההגדרה] of function $f(x)$.
- (2) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes.
- (3) Find the intervals on which function $f(x)$ is increasing and decreasing.
- (4) Sketch a graph of function $f(x)$.

Consider function $g(x)$ whose derivative function is $g'(x) = f(x) + 1$.

Function $g(x)$ and its derivative function $g'(x)$ have the same domain as function $f(x)$.

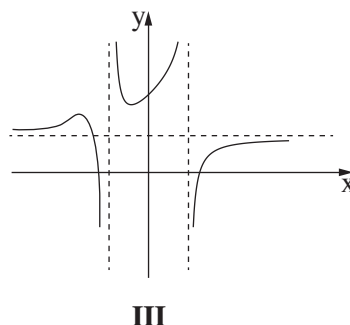
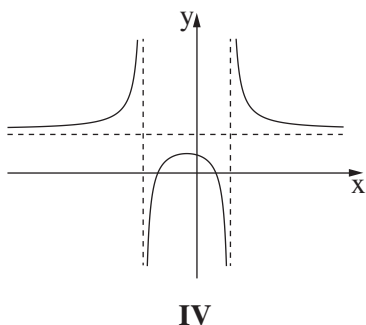
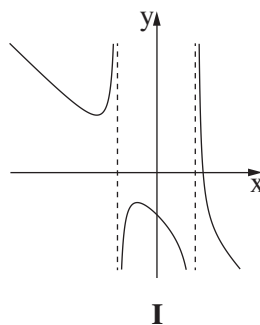
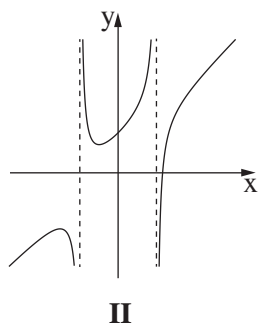
We know that function $g(x)$ has an extremum [נקודת קיצון] at a point where $x = -2$.

- ב. Find the value of a .

In function $f(x)$, substitute the value of a that you found, and answer Items ג–ד.

We know that function $g(x)$ passes through the point $(2, 7)$.

- ג. Provide a possible algebraic expression for function $g(x)$.
- ד. Determine which of the graphs I–IV below could be a graph of function $g(x)$. Explain your answer.



7. Consider the function $f(x) = 2 \cos x + \frac{1}{\cos x}$ on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- (1) Find the domain [תחום ההגדרה] of function $f(x)$.
 - (2) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the x-axis.
 - Determine whether function $f(x)$ is even or odd. Explain your answer.
 - Find the coordinates of the extrema [נקודות קיצון] of function $f(x)$, and determine what type of extrema they are.
 - Sketch a graph of function $f(x)$.

Consider the function $g(x) = 2 \cos x + \frac{2 \sin x}{\sin(2x)}$ on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

- (1) Find the domain of the function $g(x)$.
- (2) Show that $g(x) = 2 \cos x + \frac{1}{\cos x}$ for any x in its domain.
- Is there a value of k for which the line $y = k$ intersects the graph of function $g(x)$ at exactly 3 points? If there is one, find it. If not, explain why not.

8. Consider the graph of the function $f(x) = \sqrt{R^2 - x^2}$ shown in the figure on the right.

R is a positive parameter.

Function $f(x)$ is defined on the domain $-R \leq x \leq R$.

Point A is located on the graph of function $f(x)$ in the first quadrant.

A line parallel to the x-axis was drawn through Point A , intersecting the graph of function $f(x)$ at an additional point, B .

Points C and D are located on the x-axis such that a rectangle, $ABCD$, is formed.

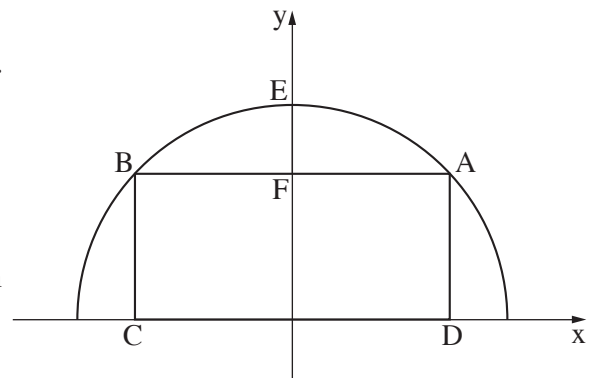
Point F is the intersection point of Side AB with the y-axis.

Point E is the intersection point of the graph of function $f(x)$ with the y-axis.

Consider a given square whose side length is equal to the length of Segment EF .

Let t be the x-coordinate of Point A .

- (1) Express the length of Segment EF in terms of R and t .
- (2) Express the perimeter of Rectangle $ABCD$ in terms of R and t .
- Express in terms of R the value of t for which the difference between the perimeter of Rectangle $ABCD$ and the perimeter of the square is greatest (maximum).



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך