

תוכנית חדשה

Note: This exam has special instructions.
Answer the questions according to these instructions.

שימו לב: בבחינה זו יש הנחיות מיוחדות.
יש לענות על השאלות על פי הנחיות אלה.

Mathematics

מתמטיקה

5 Points – 1st Exam

5 יחידות לימוד - שאלון ראשון

Instructions

הוראות

א. Duration of the exam: Four and a quarter hours.

א. משך הבחינה: ארבע שעות ורבע.

ב. Exam structure and breakdown of points:

ב. מבנה השאלון ומפתח ההערכה:

This exam has four parts and a total of eight questions.

בשאלון זה ארבעה פרקים, ובהם שמונה שאלות.

Part One: Short questions

פרק ראשון – שאלות קצרות

Part Two: Induction, Sequences, and Probability

פרק שני – אינדוקציה, סדרות והסתברות

Part Three: Planar Geometry and Planar Trigonometry

פרק שלישי – גאומטריה וטריגונומטריה במישור

Part Four: Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

פרק רביעי – חשבון דיפרנציאלי ואינטגרלי של פולינומים, של פונקציות שורש, של פונקציות רציונליות ושל פונקציות טריגונומטריות

Answer five questions, one question at least from Part One or Part Two, and at least one question from each of Part Three and Part Four.

יש לענות על חמש שאלות, על שאלה אחת לפחות מן הפרק הראשון או השני ועל שאלה אחת לפחות מכל אחד מן הפרקים השלישי והרביעי.
 $20 \times 5 = 100$ נקודות.

$5 \times 20 = 100$ points.

ג. Material that may be used during the exam:

ג. חומר עזר מותר בשימוש:

(1) A non-graphing calculator.

(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון שיט בו אפשרות תכנות.

Do not use the programming options of a programmable calculator.

שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.

Use of a graphing calculator or programming options may lead to the disqualification of your exam.

(2) דפי נוסחאות (מצורפים).

(2) Formula sheets (attached).

(3) מילון עברי-לועזי/לועזי-עברי.

(3) A Hebrew-foreign language/foreign language-Hebrew dictionary.

ד. Special instructions:

ד. הוראות מיוחדות:

(1) Do not copy the question, but do indicate the number of the question you are answering.

(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.

(2) Start each question on a new page.

(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.

Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator.

יש להסביר את כל פעולותיך, כולל חישובים, בפירוט ובצורה ברורה ומסודרת.

Explain all your work, including calculations, in detail and in a clear and organized fashion.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

Lack of detail may lower your score or may lead to the disqualification of your exam.

Write only in the answer booklet. Write "טייטה" at the top of each draft page.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טייטה" בראש כל עמוד

If you write draft material on paper outside the answer booklet, your exam may be disqualified.

המשמש טייטה.
כתיבת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Answer five of the questions 1–8, at least one question from Part One or Part Two, and at least one question from each of Part Three and Part Four. (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Short Questions

1. Answer two of the four items א–ג below. If you answer more than two items, only the first two answers in your answer booklet will be graded.

א. We know that the equality $8 + 24 + 48 + \dots + n(n+2) = \frac{n(n+a)(n+4)}{6}$ is satisfied for $n = 2$.
 a is a parameter.

(1) Find the value of a .

(2) For the value of a that you found, prove by induction or any other method that the given equality is satisfied for all even natural n .

ב. All university applicants have to take two exams.

The number of applicants who passed only the first exam was found to be 4 times smaller than the number of applicants who passed both exams.

$\frac{8}{9}$ of the applicants who passed the second exam also passed the first exam.

Let p denote the probability of randomly selecting an applicant who passed only the first exam.

(1) Express in terms of p the probability of randomly selecting an applicant who passed at least one exam.

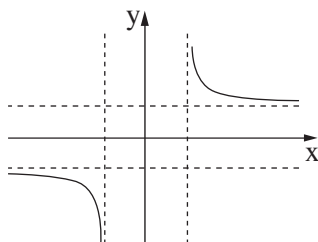
(2) We know that an applicant who passed at least one exam was randomly selected.

What is the probability that the selected applicant passed both exams?

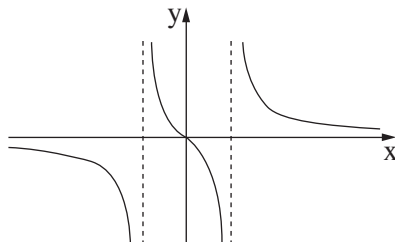
ג. Consider the functions $f(x) = \frac{x}{\sqrt{x^2 - 1}}$, $g(x) = \frac{\sqrt{x}}{x^2 - 1}$, $h(x) = \sqrt{\frac{x}{x^2 - 1}}$.

(1) Find the domain [תחום ההגדרה] of each of the functions $f(x)$, $g(x)$, and $h(x)$.

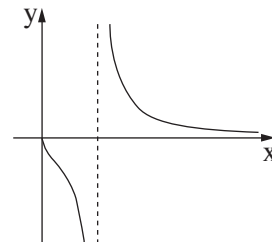
(2) Consider the graphs 1–5 below. Match each of the functions $f(x)$, $g(x)$ and $h(x)$ with the graph that describes it. Explain your answers.



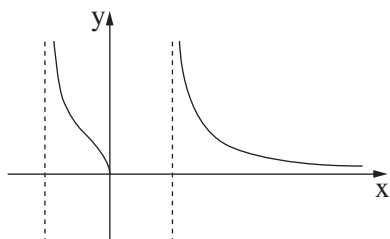
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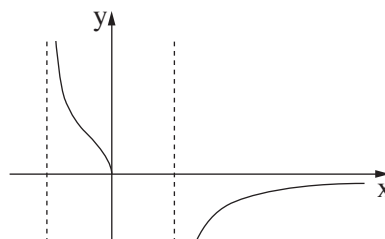
2



1



5



4

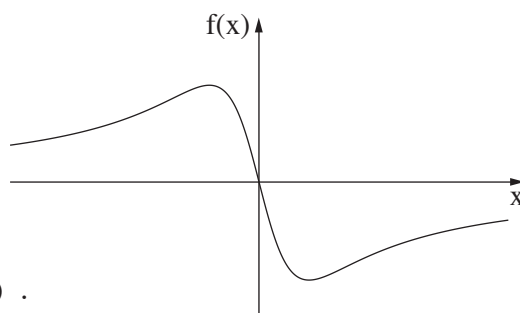
ז. The function $f(x)$ and its derivative function $f'(x)$ are defined for all x .

The figure to the right describes the graph of function $f(x)$.

We know that function $f(x)$ is odd and has only one minimum whose coordinates are $(3, -6)$.

Both function $f(x)$ and its derivative function $f'(x)$ have a horizontal asymptote $y = 0$.

(1) Sketch a possible graph of the derivative function $f'(x)$.



Consider the function $g(x) = f^2(x) \cdot f'(x)$.

(2) Find the coordinates of the intersection points of the graph of function $g(x)$ with the x -axis.

(3) Calculate the area of the region bounded by the graph of function $g(x)$ and the x -axis.

Part Two – Induction, Sequences, and Probability

2. Consider a geometric sequence A of m terms $a_1, a_2, a_3, \dots$ (m is a natural number larger than 4).

We know that: all the terms in Sequence A are negative;

the sum of the last $m - 4$ terms in the sequence is 16 times the sum of the first $m - 4$ terms in the sequence.

- א. (1) Find the common ratio of Sequence A .
(2) Is Sequence A increasing, decreasing, or neither increasing nor decreasing? Explain your answer.

Sequence A was extended to form an infinite geometric sequence.

Consider the infinite sequence B whose terms satisfy $b_n = \frac{k^n}{a_n}$ for all natural n . k is a non-zero parameter.

- א. Prove that Sequence B is a geometric sequence, and express its common ratio in terms of k .

We know that the sum of Sequence B converges.

- א. Find the possible range of values of k .

We know that: the common ratio of Sequence B is $\frac{1}{4}$;
the sum of sequence B is -9 .

- א. Find the value of k and the value of b_1 .

In Sequence B , every third term ($b_3, b_6, b_9 \dots$) was deleted.

- א. Find the sum of the remaining terms.

3. In Barrel 'א, there are 10 red balls and 15 yellow balls, and in Barrel 'ב there are only red balls.

Dana randomly selects a barrel and randomly removes a ball from it.

If the ball is yellow, she removes another random ball from the same barrel (without replacement).

If the first ball is red, she replaces the ball in the barrel and again removes a random ball from the same barrel.

- א. We know that Dana removed two balls of the same color. What is the probability that both balls are yellow?

Dana replaces the balls she had removed in the barrel.

Yael proceeds as follows:

She selects a random barrel, removes a random ball and replaces it in the barrel.

Yael repeats this process until she gets a red ball, at which point she replaces the ball in the barrel and stops removing balls.

- א. Find the probability that Yael performed this process exactly 6 times.

Some balls from Barrel 'ב were transferred to Barrel 'א.

Next, a random barrel was selected, and one ball was randomly removed from it.

We know that, after the transfer, the probability that this ball was red was $\frac{19}{24}$.

- א. Is it possible that before the transfer there were 10 balls in Barrel 'ב? Explain your answer.

Part Three – Planar Geometry and Planar Trigonometry

4. In Quadrilateral BKCD, Side KB is parallel to Side CD.

Side CD is a chord in a circle, and Side KB is tangent to the circle at Point K.

Side BD intersects the circle at Point A.

Diagonal BC intersects Chord AK at Point E (see figure).

א. Prove that $\triangle ABK \sim \triangle AKC$.

ב. Prove that $\frac{AB}{AC} = \frac{BE}{CE}$.

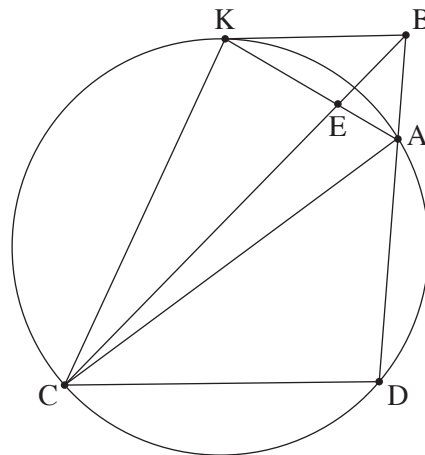
We know that $BE = \frac{1}{4}CE$.

ג. Find the ratio $\frac{AK}{AB}$.

We know that the area of Quadrilateral ABKC is 18.

Let S denote the area of Triangle AEB.

ד. Express the area of Triangle KEC in terms of S.



5. Consider a circle of center O and radius R.

Two lines, AB and AC, are drawn from Point A, such that they are tangent to the circle.

Point E is the point of tangency of Line AC with the circle, as shown in the figure.

We know that $BO \perp AO$, $AE = CE$.

Let 2β denote the angle BAC.

א. Express the length of AB in terms of R and β .

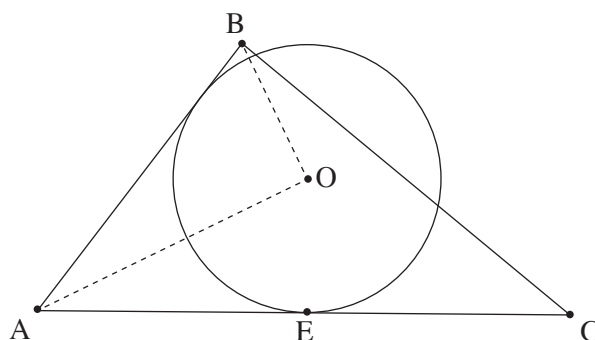
We know that the length of AB is $2.5R$ and that Angle BAC is an acute angle.

ב. Find the value of β .

ג. Find the ratio between the area of Triangle ABC and the area of Triangle AOB.

We know that the length of the radius of the circle that circumscribes Triangle ABC is 8.

ד. Find the value of R.



Part Four – Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. We know that $f'(x) = \frac{-4x}{(x^2 - a^2)^3}$ is the derivative function of function $f(x)$. a is a positive parameter.

Functions $f(x)$ and $f'(x)$ are defined on the same domain.

Express your answers in Items א–ג in terms of a , if necessary.

- א. Find the domain [תחום ההגדרה] of the derivative function $f'(x)$.
ב. Find the increasing and decreasing intervals of function $f(x)$.

We know that function $f(x)$ has a horizontal asymptote whose equation is $y = 0$.

- ג. Find a function $f(x)$ which satisfies these conditions.

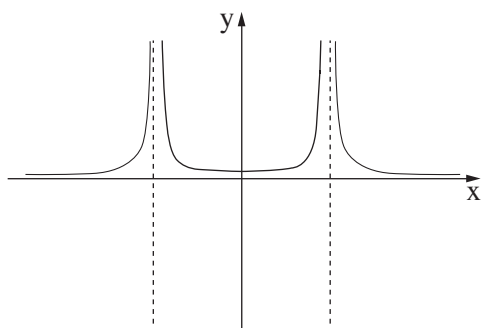
Consider function $g(x) = \frac{x - a}{(x^2 - a^2)^2}$ and function $h(x) = \frac{(x - a)^2}{(x^2 - a^2)^2}$.

Functions $f(x)$, $g(x)$, and $h(x)$ are defined on the same domain.

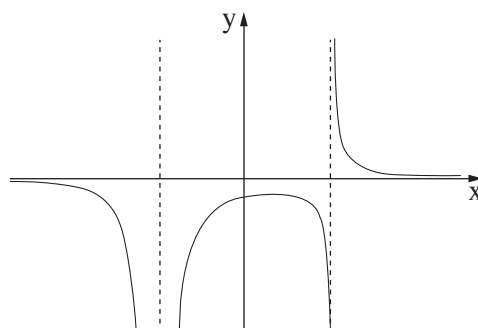
- ד. Match each of the functions $f(x)$, $g(x)$, and $h(x)$ with a graph that could represent it from the graphs I–IV at the end of this question. Explain your answers.

We know that the function $h(x - 6)$ has an asymptote whose equation is $x = 0.25a$.

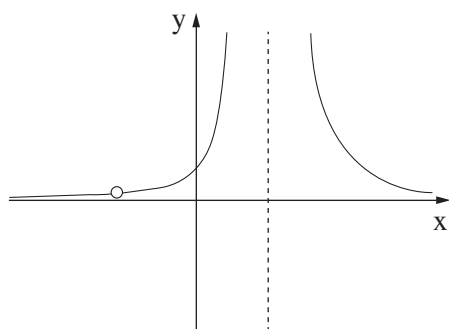
- ה. Find the value of a .



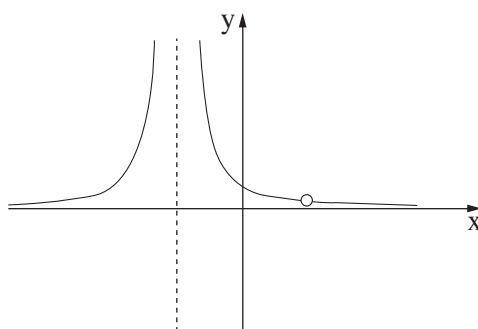
II



I



IV



III

7. Consider the function $f(x) = \cos x + \frac{2}{(\cos x)^2} + a$, on the interval $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$.
 a is a parameter.
- Find the domain [תחום ההגדרה] of function $f(x)$.
 - Is function $f(x)$ even or odd? Explain your answer.

We know that the graph of function $f(x)$ is tangent to the line $y = 2$ at two points.

- Find the value of a .

Substitute $a = 1$ and answer Items 7–8.

- Find the coordinates of the extrema [נקודות קיצון] of function $f(x)$ and determine what type of extrema they are.
- Sketch a graph of function $f(x)$.

Consider function $g(x) = f(x) - k$ and function $h(x) = \frac{1}{g(x)}$, both on the interval $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$.
 k is a parameter, $k \neq 2$, $k \neq 4$.

- Find the value of k for which the graphs of function $g(x)$ and of function $h(x)$ meet at each of their extrema.

8. Consider the function $f(x) = x\sqrt{21-x}$.

- (1) Find the domain [תחום ההגדרה] of function $f(x)$.
- (2) Find the positive and negative intervals of function $f(x)$.

We know that function $f(x)$ has one local extremum [נקודת קיצון פנימית] and no inflection points [נקודות פיתול].

- Sketch a graph of function $f(x)$.

Point A is located in the first quadrant on the graph of function $f(x)$.

Point C is the intersection point of the graph of function $f(x)$ with the positive side of the x-axis.

Two perpendiculars were dropped from Point A :

One perpendicular was dropped to the x-axis intersecting it at Point B, and another perpendicular was dropped to line $x = 21$ intersecting the line at Point D.

- Find the coordinates of Point A for which the perimeter of Rectangle ABCD is maximum.

Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך