

Note: There are special instructions in this exam. Please answer the questions according to these instructions.

Mathematics
5 Points — 1st Exam

Instructions

- א. Duration of the exam: Four and a quarter hours
- ב. Exam structure and breakdown of points:
This exam has three parts with a total of eight questions.
Part One: Algebra and Probability
Part Two: Planar Geometry and Planar Trigonometry
Part Three: Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions
You must answer four questions of your choice, at least one question from each part – $4 \times 25 = 100$ points.
- ג. Material that may be used during the exam:
(1) A non-graphing calculator. You may not use the programming options of a programmable calculator. Use of a graphing calculator or programming options may lead to the disqualification of your exam.
(2) Formula sheets (attached).
(3) A Hebrew–foreign language/foreign language–Hebrew dictionary.
- ד. Special instructions:
(1) Do not copy the question, but do indicate the number of the question you are answering.
(2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator. Explain all of your work, including calculations, in detail and in a clear and organized fashion. Lack of detail may lower your score or may lead to the disqualification of your exam.

Write only in the answer booklet. Write "טייטה" at the top of each draft page. If you use draft paper outside the answer booklet, your exam may be disqualified.

Good Luck!

שימו לב: בבחינה זו יש הנחיות מיוחדות.
יש לענות על השאלות על פי הנחיות אלה.

מתמטיקה
5 יחידות לימוד – שאלון ראשון

הוראות

- א. משך הבחינה: ארבע שעות ורבע.
- ב. מבנה השאלון ומפתח ההערכה:
בשאלון זה שלושה פרקים, ובהם שמונה שאלות.
פרק ראשון: אלגברה והסתברות
פרק שני: גאומטריה וטריגונומטריה במישור
פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים, של פונקציות שורש, של פונקציות רציונליות ושל פונקציות טריגונומטריות
יש לענות על ארבע שאלות, לפחות על שאלה אחת מכל פרק – $4 \times 25 = 100$ נקודות.
- ג. חומר עזר מותר בשימוש:
(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון שיש בו אפשרות תכנות.
שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.
(2) דפי נוסחאות (מצורפים).
(3) מילון עברי–לועזי/לועזי–עברי.
- ד. הוראות מיוחדות:
(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.
(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון. יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה ברורה ומסודרת.
חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טייטה" בראש כל עמוד המשמש טייטה. כתיבת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

בהצלחה!

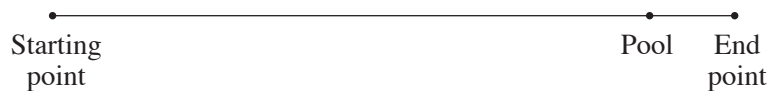
Questions

Answer four of the questions 1–8, at least one question from each part (each question – 25 points).

Note: If you answer more than four questions, only the first four answers in your answer booklet will be graded.

Part One — Algebra and Probability

1. Nadav and Yossi both set out at 8:00 on a walk along a certain route (see diagram).



Yossi walked from the starting point to the end point at a constant speed, and did not stop on the way.

Nadav walked for 1.8 km at a constant speed from the starting point, and then decided to walk back to the starting point. On his way back, Nadav walked 1.5 times faster than he had walked at first.

When Nadav arrived back at the starting point, Yossi was 4.8 km from the starting point.

Let v km/h be Nadav's initial walking speed.

- א. Express Yossi's walking speed in terms of v .

There is a pool 3.5 km from the end point.

When Nadav arrived back at the starting point, he set out again on the same route as before.

He walked at a speed of $(v + 2)$ km/h for t hours, until he arrived at the pool.

When Nadav arrived at the pool, Yossi arrived at the end point.

- ב. Express t in terms of v .

Nadav arrived at the pool at 15:30.

- ג. Find v , if we know that v is greater than 1 km/h.

2. Consider a geometric sequence of $2n + 1$ terms (n is a natural number).

All the terms in this sequence $a_1, a_2, a_3, \dots, a_{2n+1}$ are positive.

The sum of the sequence's terms excluding the first two terms is 4 times greater than the sum of the sequence's terms excluding the last two terms.

We know that the sum of the terms after the middle term is 256 times greater than the sum of the terms before the middle term.

א. Find n .

The given sequence is extended so that an infinite geometric sequence is formed.

Given: Sequence B is an infinite sequence satisfying $b_k = \frac{1}{(a_k + a_{k+1})^2}$ for all natural k .

ב. Prove that Sequence B is a geometric sequence, and find its common ratio.

All terms of Sequence B in the even positions are multiplied by 2.

We know that after this multiplication, the sum of the terms in the odd positions is greater than the sum of the terms in the even positions by $\frac{1}{30}$.

ג. Find a_1 .

3. A study was conducted among the students of a certain school to investigate the association between membership in a youth movement and volunteering in the community.

80% of the members of youth movements volunteer in their community.

Among the students who are members of a youth movement, 5 are selected at random (drawing with replacement).

א. What is the probability that at least one of the selected students is a volunteer and at least one is not a volunteer?

We know that 55% of the students are not members of a youth movement and do not volunteer, and that $\frac{1}{12}$ of the students who are not members of a youth movement do volunteer.

ב. What percentage of the students are members of a youth movement?

A total of 100 students participated in this study.

ג. How many of the students are members of a youth movement and do not volunteer?

ד. Among the students who do not volunteer, 3 are selected at random (drawing without replacement).

- (1) What is the probability that the first student who was selected is a member of a youth movement and that the next two students selected are not members of a youth movement?
- (2) What is the probability that one of the students who were selected is a member of a youth movement and the other two are not, if we know that the first student who was selected is not a member of a youth movement?

Part Two — Planar Geometry and Planar Trigonometry

4. Triangle ABC is inscribed in a circle of center O such that AB is a diameter in the circle.

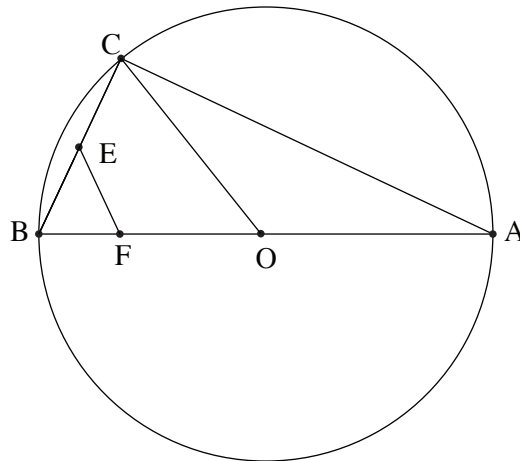
Point E is located on Side BC ,

and Point F is located on Segment BO , as shown in

the figure.

We know that Quadrilateral $CEFO$ can be inscribed in a circle.

א. Prove that $EF = EB$.



The circle that circumscribes Quadrilateral $CEFO$ intersects Side AC at Point D such that ED is parallel to AB .

א. (1) Prove that Quadrilateral $EDOB$ is a parallelogram.

(2) Prove that $OD \perp AC$.

The line ℓ is tangent at Point C to the circle circumscribing Triangle ABC .

א. Prove that the line ℓ is tangent to the circle that circumscribes Quadrilateral $CEFO$.

5. The acute triangle ABC is inscribed in a circle of center O and radius R .

Point D is the midpoint of the minor arc BC , as shown in the figure.

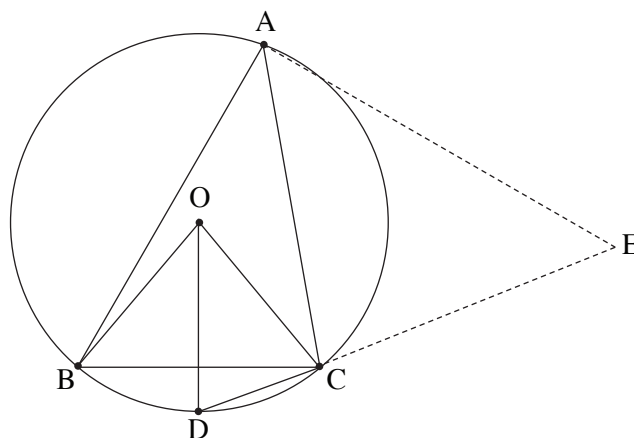
We know that $\angle ABC = 60^\circ$.

Let $\angle BAC = \alpha$.

א. Express the area of Triangles ABC and ODC in terms of R and α .

We know that the ratio of the area of Triangle ABC to the area of Triangle ODC is $2\sqrt{3} \sin(80^\circ)$.

א. Find the value of α .



Point E is located on the extension of Chord DC such that $\angle CAE = 50^\circ$, as shown in the figure.

א. Express the radius of the circle inscribed in Triangle ACE in terms of R .

Part Three— Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider function $f(x) = \frac{x-1}{(x-a)^3}$ defined on the domain [תחום ההגדרה] $x \neq a$, a is a non-zero parameter.
- א. Answer Sub-Items (1)–(2). Express your answers in terms of a , if necessary.
- (1) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes.
 - (2) Find the coordinates of the intersection point of the graph of function $f(x)$ with the y -axis.
- ב. Find for which values of a function $f(x)$ has an extremum [נקודת קיצון] positioned to the left of the asymptote that is perpendicular to the x -axis, and determine what type of extremum it is.

Graphs 1–4 shown at the end of this question describe function $f(x)$ for different values of a .

- ג. Match each of the values of a (I–IV) with the correct graph, and explain your answers.

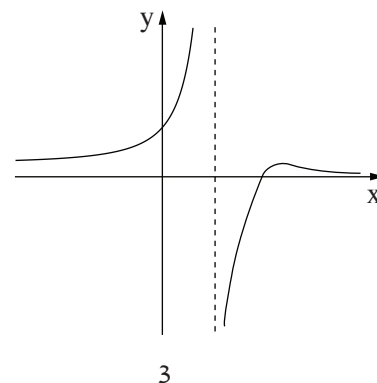
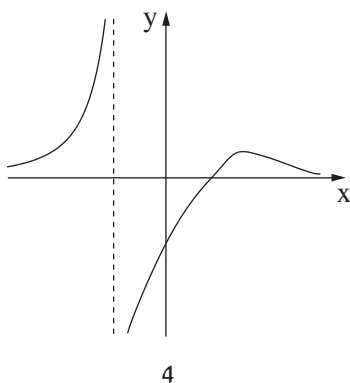
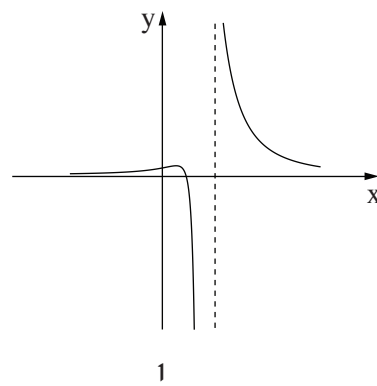
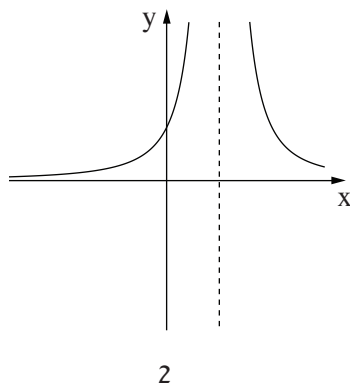
I. $a = -1$ II. $a = 0.5$ III. $a = 1$ IV. $a = 2$

Consider function $g(x)$, which satisfies $g(x) = f(x) - b$ for the value of a that matches Graph 2. b is a positive parameter.

$(t, 0)$, $1 < t < 5$, is one of the intersection points of the graph of function $g(x)$ with the x -axis.

We know that the area bounded by the graph of function $g(x)$, the horizontal asymptote of function $g(x)$, the line $x = t$, and the line $x = 5$ is 1.75.

- ד. Find the value of b .



7. Consider function $f(x) = (b + \cos x) \sin x$, defined on the domain $[-\pi \leq x \leq \pi]$, b is a parameter.

א. Is function $f(x)$ even or odd? Explain your answer.

We know that the graph of function $f(x)$ has exactly three points of intersection with the x-axis.

ב. Consider the following three possible values of b , I–III.

Determine which possibility might match function $f(x)$, and explain your answer.

I. $b = 0$ II. $0 < b < 1$ III. $1 \leq b$

We know the tangent to the graph of function $f(x)$, where $\cos x = \frac{1}{4}$, has a slope of $\left(-\frac{5}{8}\right)$.

ג. Find the value of b .

Substitute $b = 1$ in function $f(x)$, and answer Items ד–ו.

ד. Find the coordinates of the extrema [נקודות קיצון] of function $f(x)$ and determine what type of extrema they are.

ה. (1) Sketch a graph of function $f(x)$.

(2) Sketch a graph of the derivative function $f'(x)$.

Consider function $g(x)$ which satisfies $g(x) = (f(x))^2 \cdot f'(x)$. Function $g(x)$ is defined on the domain $0 \leq x \leq \pi$.

ו. Calculate the area in the first quadrant that is bounded by the graph of function $g(x)$ and the x-axis.

8. Consider the function $f(x) = \sqrt{x}$, defined on the domain $[x \geq 0]$, and the function $g(x) = \frac{16}{x^2 + 3}$ defined for all x .

א. (1) Find the coordinates of the extremum [נקודת הקיצון] of function $g(x)$, and determine what type of extremum it is.

(2) Find the coordinates of the inflection points [נקודות הפיתול] of function $g(x)$.

(3) Sketch graphs of the functions $f(x)$ and $g(x)$ on the same coordinate system.

A perpendicular to the x-axis is drawn through Point $C(t, 0)$, $t > 0$.

The perpendicular intersects the graph of function $f(x)$ at Point A and the graph of function $g(x)$ at Point B.

ב. Express the product of the lengths of Segments AC and BC in terms of t .

ג. Prove that the product of the lengths of Segments AC and BC is a maximum when Point B is an inflection point of function $g(x)$.

Consider the function $k(x) = \frac{8\sqrt{x-4}}{(x-4)^2 + 3}$, defined on the domain $x \geq 4$.

ד. Use the previous items of this question to find the coordinates of the local extremum [נקודת הקיצון הפנימית] of function $k(x)$, and determine what type of extremum it is. Explain your answer.

Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך