

Mathematics
5 Points — 1st Exam

Instructions

- א. Duration of the exam: Three and a half hours
- ב. Exam structure and breakdown of points:
This exam has three parts with a total of eight questions.
Part One: Algebra and Probability
Part Two: Planar Geometry and Planar Trigonometry
Part Three: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions
You must answer five questions of your choice, at least
one question from each part – $5 \times 20 = 100$ points.
- ג. Material that may be used during the exam:
(1) A non-graphing calculator. You may not use the
programming options of a programmable calculator.
Use of a graphing calculator or programming options
may lead to the disqualification of your exam.
(2) Formula sheets (attached).
(3) A Hebrew–foreign language/foreign language–Hebrew
dictionary.
- ד. Special instructions:
(1) Do not copy the question, but do indicate the number
of the question you are answering.
(2) Start each question on a new page. Write the steps
of the solution in the answer booklet, even when the
calculations are done with a calculator.
Explain all of your work, including calculations, in
detail and in a clear and organized fashion. Lack
of detail may lower your score or may lead to the
disqualification of your exam.

Write only in the answer booklet. Write "טיוטה" at the top of each
draft page. If you use draft paper outside the answer booklet, your
exam may be disqualified.

Good Luck!

מתמטיקה
5 יחידות לימוד – שאלון ראשון

הוראות

- א. משך הבחינה: שלוש שעות וחצי.
- ב. מבנה השאלון ומפתח ההערכה:
בשאלון זה שלושה פרקים, ובהם שמונה שאלות.
פרק ראשון: אלגברה והסתברות
פרק שני: גאומטריה וטריגונומטריה במישור
פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,
של פונקציות שורש, של פונקציות רציונליות
ושל פונקציות טריגונומטריות
יש לענות על חמש שאלות, לפחות על שאלה אחת מכל פרק
 $100 = 20 \times 5$ נקודות.
- ג. חומר עזר מותר בשימוש:
(1) מחשבון לא גרפי.
אין להשתמש באפשרויות התכנות במחשבון שיש בו אפשרות
תכנות.
שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול
לגרום לפסילת הבחינה.
(2) דפי נוסחאות (מצורפים).
(3) מילון עברי–לועזי/לועזי–עברי.
- ד. הוראות מיוחדות:
(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.
(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את
שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.
יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה
ברורה ומסודרת.
חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טיוטה" בראש כל עמוד המשמש
טיוטה. כתיבת טיוטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת
הבחינה.

בהצלחה!

Questions

Answer five of the questions 1–8, at least one question from each part (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Algebra and Probability

1. Point A is located north of Point O, and Point B is located south of Point O.

Point C is located 15 km to the east of Point O, as shown in the figure.

On Sunday, Ori went out for a walk from Point O to Point A.

At the same time, Smadar went for a run from Point C in the direction of Point O.

Smadar's running speed was 3 times greater than Ori's walking speed.

We know that the aerial distance between Ori and Smadar at the moment

Ori arrived at Point A was $\sqrt{145}$ km.

Ori and Smadar were moving at constant speeds.

- א. Find the distance that Ori walked and the distance that Smadar ran on Sunday, if we know that Smadar passed Point O.

On the same day, Boaz went out for a walk from Point O in the direction of Point B. Boaz set out on his walk 20 minutes after Ori had set out on his walk.

Boaz walked at a constant speed that was 50% faster than Ori's walking speed.

When Ori reached Point A, the distance between him and Boaz was 18 km, and they both stopped walking at that moment.

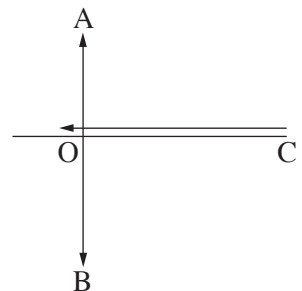
- ב. Find Ori's walking speed and Boaz's walking speed.

On Monday, Ori and Boaz went out for a walk at the same time. Each set out from the point at which he had stopped on Sunday, and each walked in the same direction as on Sunday. Boaz

reduced his walking speed by v km/h and Ori increased his walking speed by v km/h.

They both stopped when the distance between them was 24 km.

- ג. Find how many minutes Ori walked on Monday.



2. Consider an arithmetic sequence $a_1, a_2, \dots, a_{3n}$ of $3n$ terms, whose common difference is d .

Let S_n^* be the sum of the n middle terms of the sequence.

א. Prove that $S_n^* = \frac{1}{3} \cdot S_{3n}$.

We know that the first term in the sequence is positive and that the sum of the n middle terms is 0.

- ב. Is the common difference of this sequence positive or negative? Explain your answer.

We know that $a_1 = 19 \cdot |d|$ is satisfied.

- ג. Find the number of terms in the sequence.

We delete some of the terms in the given sequence, and a new arithmetic sequence is created:

$$a_2, a_5, a_8, \dots, a_{3n-4}.$$

The sum of the terms in the new sequence is 36.

- ד. Find d .

3. A daily newspaper is delivered to subscribers who live in either Haifa or Tel Aviv. The newspaper is supposed to arrive at their homes every day by 6:00 am.

The subscription department polled subscribers, and asked whether they had received the newspaper on time on a specific day.

All subscribers participated in the poll, and each one answered either yes or no.

The results of the poll showed that the probability of randomly selecting a subscriber who had received the newspaper on time from among the subscribers from Haifa is $\frac{3}{4}$, and the probability of randomly selecting a subscriber from Haifa from among the subscribers who had received the newspaper on time is $\frac{5}{9}$.

Let p be the probability that a subscriber selected at random from among all subscribers lives in Haifa.

A subscriber is selected at random.

- א. Express in terms of p the probability that the selected subscriber lives in Tel Aviv and got the newspaper on time.

We know that the number of subscribers who live in Tel Aviv and did not get their newspaper on time is 1.5 times greater than the number of subscribers who live in Tel Aviv and got their newspaper on time.

- ב. What percentage of the subscribers got their newspaper on time?

Of the subscribers who did not get their newspaper on time, two are selected at random.

- ג. What is the probability that the first one selected lives in Tel Aviv and the second lives in Haifa?

On the same day, 6 subscribers called the subscription department to say that they had not received their newspaper on time.

- ד. What is the probability that, at most, 4 of them live in Haifa?

Part Two — Planar Geometry and Planar Trigonometry

4. Two lines are drawn from Point B, which is outside a circle: one is tangent to the circle at Point C and the other intersects the circle at Points E and A, as shown in the figure.

Point D is located on the circle, such that Chord CD is parallel to Chord EA.

Chords ED and AC intersect at Point K.

- א. Prove that $\triangle CEB \sim \triangle DCE$.

We know that $AK = 3$ and $ED = 7$.

Let S be the area of Triangle CEK.

- ב. Express the area of Triangle CKD in terms of S.

We know that $BC = \frac{35}{\sqrt{32}}$.

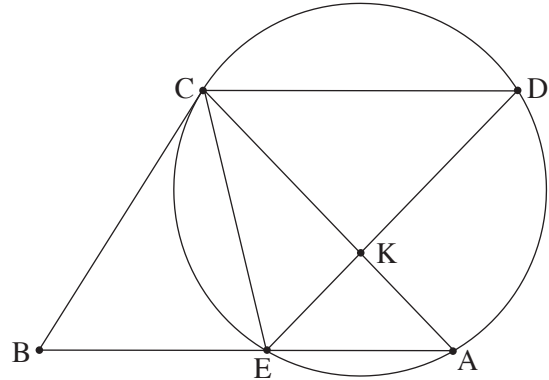
- ג. Express the area of Triangle CEB in terms of S.

Point O is the center of the circle.

- ד. Prove that $\angle COE = \angle CKE$.

We know that $\angle CAE = 45^\circ$.

- ה. Explain why Points O, C, E, and K are located on one circle.



5. Consider a trapezoid ABCD ($AB \parallel DC$) which is inscribed in a circle.

The extensions of Sides AD and BC meet at Point E, as shown in the figure.

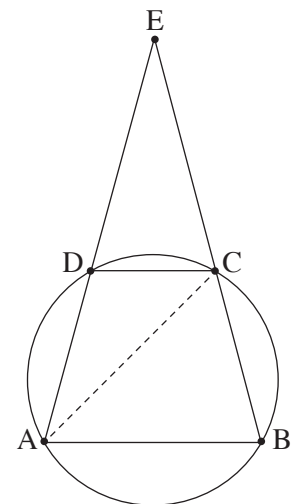
We know that $\angle ACB = 60^\circ$.

Let $AC = k$, $\angle CDE = \alpha$.

- א. (1) Find the angles of Triangle ACE (express in terms of α , if necessary).
(2) Express the length of Sides AB and DC in terms of α and k.

We know that the area of Triangle ABE is 3 times greater than the area of Triangle DCE.

- ב. Find the size of the angle α .
ג. Find the value of k for which the length of the median to Side EC in Triangle AEC is $\sqrt{7}$.



Part Three— Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the function $f(x) = \frac{x^2 - a^2}{(x - 4)^2}$, $0 < a < 4$ is a parameter.
- א. Answer Sub-Items (1)–(5). Express your answers in terms of a , if necessary.
- (1) Find the domain [תחום ההגדרה] of function $f(x)$.
 - (2) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes.
 - (3) Find the coordinates of the intersection points of the graph of function $f(x)$ with the axes.
 - (4) Find the x -coordinate of the extremum [נקודת הקיצון] of function $f(x)$, and determine what type of extremum it is.
 - (5) Sketch a graph of function $f(x)$.

Consider the function $g(x) = \frac{x^2}{(x - 4)^2}$, which is defined on the same domain as function $f(x)$.

- ב. (1) Prove that the graph of function $g(x)$ is entirely located above the graph of function $f(x)$.
- (2) Express in terms of a the area bounded by the graphs of functions $f(x)$ and $g(x)$, the line $x = 1$, and the y -axis.

7. Consider the function $f(x) = \frac{2x}{\sqrt{x^2 + x}}$.

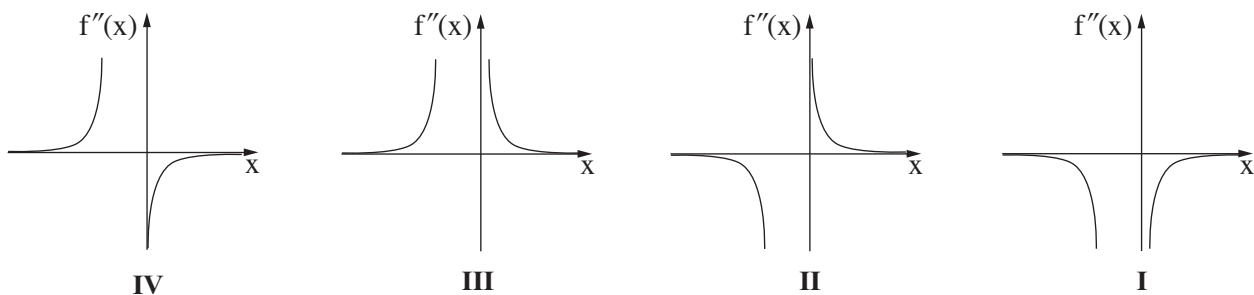
- א. (1) Find the domain [תחום ההגדרה] of function $f(x)$.
- (2) Does the graph of function $f(x)$ intersect the axes? Explain your answer.
- (3) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes.
- (4) Find the intervals on which function $f(x)$ is increasing and decreasing.

We know that function $f(x)$ has no inflection points [נקודות פיתול].

- ב. Sketch a graph of function $f(x)$.
- ג. Use the graph of function $f(x)$ to determine which of the graphs I–IV at the end of the question describes the graph of the second derivative $f''(x)$.

Explain your answer.

- ד. Calculate the area of the region bounded by the graph of the second derivative function $f''(x)$, the x-axis, and the lines $x = 1$ and $x = 2$.



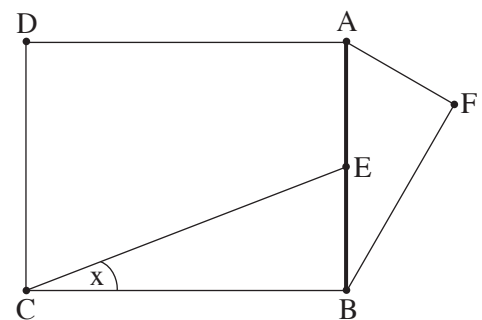
8. Point E is the midpoint of Segment AB.

A rectangle ABCD and a right triangle AFB, $\angle AFB = 90^\circ$, are constructed on Segment AB, as shown in the figure.

We know that $\angle ECB = x$, $\angle FAB = 2x$.

Let h be the length of Segment AB.

- א. What range of values is possible for x ? Explain your answer.
- ב. Express the difference between the length of Segment CE and the length of Segment AF in terms of x and h .
- ג. Find the value of x for which the difference between the length of Segment CE and the length of Segment AF is smallest (minimum).
- ד. For the value of x which you found in Item ג, find the ratio between the area of Rectangle ABCD and the area of Triangle AFB.



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך