

Mathematics
5 Points – 1st Exam

Instructions

- א. Duration of the exam: Three and a half hours
- ב. Exam structure and breakdown of points:
This exam has three parts, with a total of eight questions.
Part One: Algebra and Probability
Part Two: Planar Geometry and Planar Trigonometry
Part Three: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions
You must answer five questions, at least one question
from each part – $5 \times 20 = 100$ points.
- ג. Material that may be used during the exam:
(1) A non-graphing calculator. You may not use the
programming options of a programmable calculator.
Use of a graphing calculator or programming options
may lead to the disqualification of your exam.
(2) Formula sheets (attached).
(3) A Hebrew–foreign language/foreign language–
Hebrew dictionary.
- ד. Special instructions:
(1) Do not copy the question, but do indicate the number
of the question you are answering.
(2) Start each question on a new page. Write the steps
of the solution in the answer booklet, even when the
calculations are done with a calculator.
Explain all of your work, including calculations, in
detail and in a clear and organized fashion.
Lack of detail may lower your score or may lead to
the disqualification of your exam.

Write only in the answer booklet. Write "טיטה" at the top of each
draft page. If you write draft material on paper outside the answer
booklet, your exam may be disqualified.

Good Luck!

מתמטיקה
5 יחידות לימוד – שאלון ראשון

הוראות

- א. משך הבחינה: שלוש שעות וחצי.
- ב. מבנה השאלון ומפתח ההערכה:
בשאלון זה שלושה פרקים, ובהם שמונה שאלות.
פרק ראשון: אלגברה והסתברות
פרק שני: גאומטריה וטריגונומטריה במישור
פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים, של
פונקציות שורש, של פונקציות רציונליות ושל
פונקציות טריגונומטריות
יש לענות על חמש שאלות, לפחות שאלה אחת מכל פרק
– $5 \times 20 = 100$ נקודות.
- ג. חומר עזר מותר בשימוש:
(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון
שיש בו אפשרות תכנות.
שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול
לגרום לפסילת הבחינה.
(2) דפי נוסחאות (מצורפים).
(3) מילון עברי–לועזי/לועזי–עברי.
- ד. הוראות מיוחדות:
(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.
(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את
שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.
יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה
ברורה ומסודרת.
חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טיטה" בראש כל עמוד המשמש
טיטה. כתיבת טיטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת
הבחינה.

בהצלחה!

Questions

Answer five of the questions 1–8, at least one question from each part (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One – Algebra and Probability

1. Two cyclists, Cyclist a and Cyclist b, set out from two points, A and B, respectively, and cycled toward each other.

The distance between Points A and B is $3d$ km (d is a positive parameter).

Cyclist b started out 2.5 hours after Cyclist a had started out.

At 15:30 each of the cyclists had travelled one third of the distance between Points A and B.

Each of the cyclists cycled at a constant speed.

The next day, the two cyclists again set out from the same points, A and B, and cycled toward each other.

Each of the cyclists cycled at the same speed as they had cycled the first day. This time, though, they set out at the same time and met 9 hours later.

⌘. (1) At what time on the first day did Cyclist a leave Point A?

(2) Express each of the cyclists' speed in terms of d .

Cyclist a takes 1.25 more minutes to ride one kilometer than it takes Cyclist b to ride one kilometer.

- Ⓐ. Find the distance between Points A and B.

2. Consider two infinite, convergent geometric sequences, A and B , whose terms have all non-zero values.

The general term of Sequence A is a_n and its common ratio is q_A .

The general term of Sequence B is b_n and its common ratio is q_B .

A new infinite, convergent geometric sequence is constructed from the sequences A and B . Its terms are:

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots, \frac{a_n}{b_n}, \dots$$

All three sequences, Sequence A , Sequence B , and the new sequence, are not constant.

- א. Express the common ratio of the new sequence in terms of q_A and q_B .

Sequence A is neither increasing nor decreasing, and Sequence B is increasing.

- ב. Determine whether each of the two statements (1)–(2) below is true or false, and explain your answers.

- (1) The common ratio of the new sequence is positive.
(2) All the terms in Sequence B are negative.

The numbers c_1 , c_2 , and c_3 are the first three terms in an arithmetic sequence.

We know that $c_2 = -c_1$ and that $\frac{c_1 \cdot c_2}{c_3} = -\frac{1}{45}$ is satisfied.

- ג. Find c_1 .

We know that the common ratio of Sequence A equals c_1 ,

and that $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots = \frac{a_1 + a_2 + a_3 + \dots}{b_1 + b_2 + b_3 + \dots}$ is satisfied.

- ד. Find the value of q_B .

3. A proposal was made at a large college to shorten the lunch break so that the school day would end earlier.

In response, the college held a poll for all first- and second-year students.

The results of the poll showed that 80% of the participants who supported the proposal were first-year students. The results also showed that the number of first-year students who supported the proposal was the same as the number of second-year students who were against the proposal. Of the students who participated in the poll, none abstained.

Let p be the probability of randomly selecting a student who supported the proposal out of all the students who participated in the poll.

- א. A second-year student is selected at random. What is the probability that this student is against the proposal?

We know that the probability that a randomly selected first-year student supports the proposal is greater by $\frac{13}{35}$ than the probability that a randomly selected second-year student supports the proposal.

- ב. Calculate the value of p .

- ג. A participant in the poll is selected at random. Calculate the probability of at least one of these two conditions occurring:

- I The selected participant is a second-year student.
- II The selected participant supports the proposal.

- ד. Five (5) participants in the poll are selected at random.

We know that all five selected participants are second-year students.

What is the probability that at least two of them support the proposal and, also, at least two are against the proposal?

Part Two – Planar Geometry and Planar Trigonometry

4. Points A, B, and C are on a circle.

Point E is the midpoint of the arc BC, as shown in the figure on the right.

A tangent to the circle is drawn through Point E.

The tangent intersects the extension of Chord AB at Point G.

The chords AE and BC intersect at Point F.

- א. Prove that $\triangle ACE \sim \triangle AEG$.

We know that $AG = 6$ and $AE = 3\sqrt{6}$.

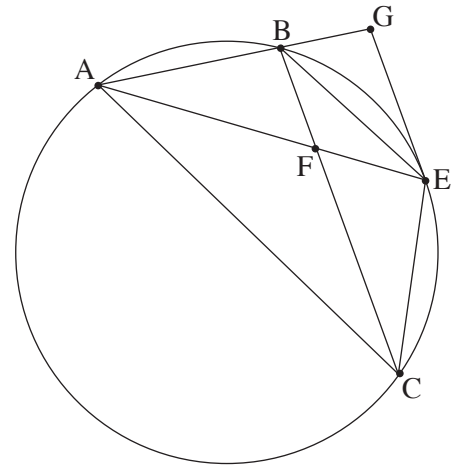
- ב. Calculate the length of Chord AC.

- ג. Prove that $BC \parallel GE$.

We know that the area of Triangle ABF is double the area of Triangle BFE.

- ד. Calculate the length of Chord AB.

- ה. What is the ratio between the area of Triangle ABF and the area of Triangle AFC? Explain your answer.



5. Kite ABCD is inscribed in a circle whose radius is R.

Chord AC is the main diagonal of the kite.

Point O is the center of the circle inscribed in Triangle ABC (see figure).

Let $\angle CAB = \alpha$.

- א. (1) Find the angles of Triangle AOC (express in terms of α , if necessary).
(2) Express the length of Segment AO in terms of α and R.

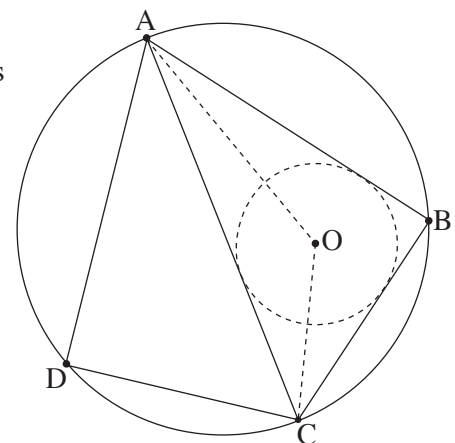
We know that the length of Segment AO is $R\sqrt{2}$.

- ב. Find the size of the angle α .

We know that the area of the kite is $25\sqrt{3}$.

- ג. Find R.

- ד. Calculate the distance between the center of the circle that circumscribes the kite and the center of the circle inscribed in Triangle ABC.



Part Three – Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the given function $f(x) = \frac{2a - x^2}{x}$, defined for $x \neq 0$; a is a positive parameter.
- א. Express your answers in terms of a , if necessary.
- (1) Find the equations of the asymptotes of function $f(x)$ that are perpendicular to the axes, if there are any.
 - (2) Show that function $f(x)$ is an odd function.
 - (3) Find the coordinates of the intersection points of the graph of function $f(x)$ with the axes, if there are any.
 - (4) Find the intervals on which function $f(x)$ is increasing and on which it is decreasing, if there are any.
 - (5) Find the concave up ($\cup$) interval and the concave down ($\cap$) interval of function $f(x)$.
- ב. Sketch a graph of function $f(x)$.

Consider another function $g(x) = |f(x)| - b$, b is a positive parameter.

Function $g(x)$ is defined in the same domain as function $f(x)$.

- ג. Sketch a graph of function $g(x)$.

We know that one of the extrema [נקודות קיצון] of function $g(x)$ is: $(2, -3)$.

- ד. Find the values of a and b .

Consider a further function $s(x) = \int_1^x g(t) dt$ defined in the domain $1 < x$.

- ה. What type of extremum does function $s(x)$ have? Explain your answer.

7. Consider function $f(x)$, which is defined in the domain $x \leq a$, $x \neq 0$. a is a positive parameter.

The figure on the right describes the graph of the derivative function $f'(x)$.

The derivative function $f'(x)$ is defined in the domain $x < a$, $x \neq 0$.

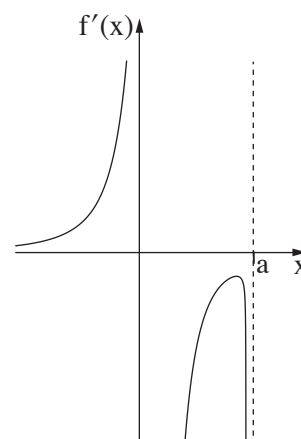
The derivative function $f'(x)$ has three asymptotes that are perpendicular to the axes and whose equations are:

$$y = 0, \quad x = a, \quad x = 0.$$

In the interval $x < 0$, the derivative function $f'(x)$ is increasing.

The line $x = 0$ is also an asymptote of the graph of function $f(x)$.

$$f(a) = 0.$$



- א. (1) Find the increasing interval and the decreasing interval of function $f(x)$ (express your answer in terms of a , if necessary). Explain.
- (2) How many inflection points [נקודות פיתול] does function $f(x)$ have? Explain.

We know that the line $y = 0$ is an asymptote of the graph of function $f(x)$.

- ב. Sketch a possible graph of function $f(x)$, based on your answer in Sub-item א(2).

We know that one of the expressions I–IV below represents function $f(x)$.

I. $\frac{\sqrt{a-x}}{x^2}$ II. $\frac{\sqrt{x-a}}{x^2}$ III. $\frac{\sqrt{a-x}}{x}$ IV. $\frac{\sqrt{x-a}}{x}$

- ג. Which of the expressions I–IV represents function $f(x)$? Explain.

We know that the slope of the tangent to the graph of function $f(x)$ at the point where $x = (-2)$, is $\frac{7}{16}$.

- ד. Find the value of a .

- ה. Substitute $a = 2$ and calculate the area of the region bounded by the graph of function $(f(x))^2$, the x -axis, and the line $x = 1$.

8. Consider the rhombus ABCD. Point E is the midpoint of Side BC.

Let $\angle ECD = x$.

We know that the area of Triangle ECD is 18.

- א. Express the side length of the rhombus in terms of x .
- ב. Calculate the minimum length of Segment DE.

Good Luck!

All rights reserved by the State of Israel.
Copying or publication without the consent of the
Ministry of Education is prohibited.

בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך