

Note: There are special instructions in this exam.

Answer the questions according to these instructions.

Mathematics
5 Points — 1st Exam

Instructions

- א. Duration of the exam: Three and a half hours
- ב. Exam structure and breakdown of points:
This exam has three parts with a total of eight questions.
Part One: Algebra and Probability
Part Two: Planar Geometry and Planar Trigonometry
Part Three: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions
You must answer five questions of your choice
— $5 \times 20 = 100$ points.
- ג. Material that may be used during the exam:
(1) A non-graphing calculator. You may not use the programming options of a programmable calculator. Use of a graphing calculator or programming options may lead to the disqualification of your exam.
(2) Formula sheets (attached).
(3) A Hebrew-foreign language/foreign language-Hebrew dictionary.
- ד. Special instructions:
(1) Do not copy the question, but do indicate the number of the question you are answering.
(2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator. Explain all of your work, including calculations, in detail and in a clear and organized fashion. Lack of detail may lower your score or may lead to the disqualification of your exam.

Write only in the answer booklet. Write "טייטה" at the top of each draft page. If you write draft material on paper outside the answer booklet, your exam may be disqualified.

Good Luck!

שימו לב: בבחינה זו יש הנחיות מיוחדות.
יש לענות על השאלות על פי הנחיות אלה!

מתמטיקה
5 יחידות לימוד - שאלון ראשון

הוראות

- א. משך הבחינה: שלוש שעות וחצי.
- ב. מבנה השאלון ומפתח ההערכה:
בשאלון זה שלושה פרקים, ובהם שמונה שאלות.
פרק ראשון: אלגברה והסתברות
פרק שני: גאומטריה וטריגונומטריה במישור
פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,
של פונקציות שורש, של פונקציות רציונליות
ושל פונקציות טריגונומטריות
יש לענות על חמש שאלות לבחירתכם
— $20 \times 5 = 100$ נקודות.
- ג. חומר עזר מותר בשימוש:
(1) מחשבון לא גרפי.
אין להשתמש באפשרויות התכנות במחשבון שיש בו אפשרות תכנות.
שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.
(2) דפי נוסחאות (מצורפים).
(3) מילון עברי-לועזי/לועזי-עברי.
- ד. הוראות מיוחדות:
(1) אין להעתיק את השאלה; יש לסמן את מספרה בלבד.
(2) יש להתחיל כל שאלה בעמוד חדש. יש לרשום במחברת את שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון. יש להסביר את כל הפעולות, כולל חישובים, בפירוט ובצורה ברורה ומסודרת.
חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

יש לכתוב במחברת הבחינה בלבד. יש לרשום "טייטה" בראש כל עמוד המשמש טייטה. כתיבת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

בהצלחה!

Questions

Note: Explain all of your work, including calculations, clearly and in detail.

Lack of detail may lower your score or lead to the disqualification of your exam.

Answer five of the questions 1-8 (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Algebra and Probability

1. Four runners participate in a relay race on a 1,440 meter track. The track is divided into 4 equal legs (sections) and one runner stands at the starting point of each leg.

At the sound of the starter pistol, the first runner sets off. As soon as he reaches the end of the first leg, the second runner sets off, and so on, until the fourth runner reaches the end of his leg.

The second runner's speed is 1.5 times greater than the first runner's speed, the third runner's speed is 2 times smaller than the second runner's speed, and the fourth runner's speed is equal to the third runner's speed. The speed of each runner is constant throughout his leg.

The four runners together completed the entire track in a total of three minutes and 54 seconds.

א. Find the running speed of each of the runners.

The third and fourth runners trained in order to increase their running speed.

Some time later, the four runners participated again in a relay race on the same track. Each one ran the same leg as in the previous race. The total running time of the third and fourth runners was 1.5 times greater than the total running time of the first two runners. The first and second runners ran at the same speed as in the previous race.

It took the third runner 2.5 seconds less to run every 100 meters than the fourth runner.

- ב. (1) By how many seconds is the third runner's running time shorter than the fourth runner's running time?

- (2) Did each of these two runners (the third and the fourth) increase his running speed?

Explain your answer.

2. Consider an infinite geometric sequence A whose general term is a_n and common ratio is q .

א. Prove that $a_1 \cdot a_{2n} = a_n \cdot a_{n+1}$ is satisfied for all natural n .

The first $2k$ terms in Sequence A satisfy the following: the product of the two middle terms in the sequence equals $10,935 \cdot a_1$.

Given: $a_{2k-2} = 1,215$.

ב. Find q (two possibilities).

Given: $a_1 = 5$.

ג. (1) Is Sequence A increasing, decreasing, or neither increasing nor decreasing?
Explain your answer.

(2) Find k .

Sequence A is used to construct an infinite sequence B as follows: $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \dots$.

ד. Prove that Sequence B is a geometric sequence.

The signs of all the terms in the odd positions in Sequence B are changed to obtain Sequence C , as follows: $-\frac{1}{a_1}, \frac{1}{a_2}, -\frac{1}{a_3}, \frac{1}{a_4}, \dots$.

ה. Find the sum of Sequence C .

3. A survey was conducted in a large city in Israel to assess the residents' proficiency in the English language.

The large number of people who participated in the survey included younger and older people.

The survey showed that the number of older people who were proficient in English was 3 times greater than the number of younger people who were proficient in English. It also showed that the number of older people who were not proficient in English was $2\frac{2}{3}$ times greater than the number of older people who were proficient in English.

Let p denote the probability of randomly selecting a younger person proficient in English from among all the survey participants.

א. What is the probability of randomly selecting an older person who is proficient in English from among all the older survey participants?

ב. Three older people are randomly selected from among the older people who participated in the survey. What is the probability that exactly two of them are proficient in English?

ג. (1) Express in terms of p the probability of randomly selecting a younger person who is not proficient in English from among all the survey participants.

(2) Show that the range of possible values of p is $0 < p < \frac{1}{12}$.

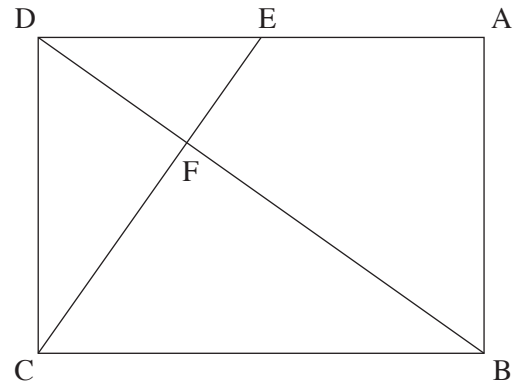
We know that the probability of randomly selecting an older person from among the survey participants who are not proficient in English is the same as the probability of randomly selecting a younger person from among the survey participants who are not proficient in English.

ד. Find the value of p .

ה. Are the events "being proficient in English" and "being an older person" dependent on each other? Explain your answer.

Part Two — Planar Geometry and Planar Trigonometry

4. In Rectangle $ABCD$, Point E is located on Side AD .
Segment CE intersects Diagonal BD at Point F .
Quadrilateral $EABF$ can be inscribed in a circle.



א. Prove that $\triangle DAB \sim \triangle BFC$.

Given: $DE = EA$.

ב. Calculate the ratio $\frac{EF}{FC}$.

Let S denote the area of Triangle DEF .

ג. Express the areas of Triangles DFC and BFC in terms of S .

ד. Calculate the similarity ratio between Triangle DAB and Triangle BFC .

Let $DE = a$.

ה. (1) Express the length of Diagonal BD in terms of a .

(2) Express the diameter of the circle circumscribing Quadrilateral $EABF$ in terms of a .

5. Consider a circle whose center is at Point O and radius is R .

A line tangent to the circle at Point D is drawn from Point A , which is outside the circle. Another line is drawn from Point A , intersecting the circle at Point B , as shown in the figure.

Let $\angle AOD = \alpha$, $\angle AOB = \beta$.

א. Express in terms of α , β and R , if necessary:

(1) the length of the segment AO .

(2) the length of the segment AB .

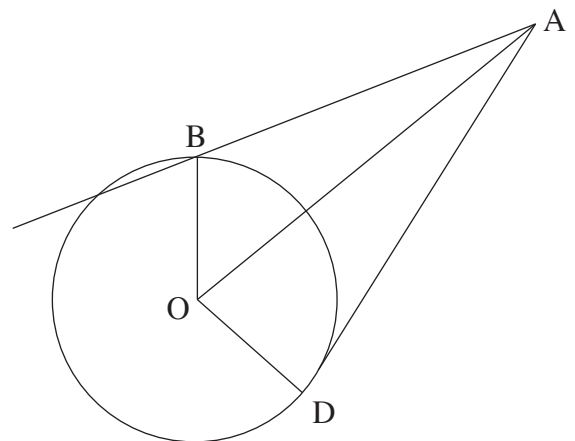
Given: $AB = \sqrt{2} R$.

ב. Prove that $\cos \beta = \frac{\sin^2 \alpha}{2 \cos \alpha}$.

Triangle ADO is inscribed in another circle whose radius is r .

Given: $\frac{R}{r} = \frac{2\sqrt{7}}{5}$.

ג. Find the size of the angles α and β .



Part Three— Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the given function $f(x) = \frac{x^2 - 9}{\sqrt{x + a}}$, a is a positive parameter.

א. Express the domain [תחום ההגדרה] of function $f(x)$ in terms of a .

We know that function $f(x)$ has no asymptotes that are perpendicular to the axes.

ב. (1) Find a .

(2) Find the coordinates of the intersection points of the graph of function $f(x)$ with the axes.

(3) Find the coordinates of the extremum [נקודת הקיצון] of function $f(x)$, and determine what type of extremum it is.

(4) Sketch a graph of function $f(x)$.

Consider the given functions $g(x) = -f(x + 2)$, $h(x) = |f(x)|$.

ג. (1) Find the domain of function $g(x)$ and the domain of function $h(x)$.

(2) Is the y-coordinate of the maximum of function $g(x)$ greater than, smaller than, or equal to the y-coordinate of the maximum of function $h(x)$? Explain your answer.

Given: $\int_{-1}^3 h(x) dx = \int_{-3}^k g(x) dx$, $k > -3$.

ד. Find k . Explain your answer.

7. Consider the given function $f(x) = \sin^2(x) - \cos^2(x) - 1$, defined for all x .

- א. Is function $f(x)$ even? Explain.
- ב. Prove that $-2 \leq f(x) \leq 0$ is satisfied for all x .
- ג. Find the coordinates of the intersection points of the graph of function $f(x)$ with the axes on the interval $-\pi \leq x \leq \pi$.
- ד. Sketch a graph of function $f(x)$ on the interval $-\pi \leq x \leq \pi$.

Consider the given function $g(x) = f(2x)$, defined for all x .

ה. Find the coordinates of the extrema [נקודות הקיצון] of function $g(x)$ on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and determine what type of extrema they are.

ו. Given: $\int_0^{\frac{\pi}{8}} (g'(x) - f'(x)) dx = S$

Express $\int_{-\frac{\pi}{8}}^0 (g'(x) - f'(x)) dx$ in terms of S . Explain your answer.

8. Consider the given function $f(x) = x^3 + 4x^2$ defined for all x .

Point B is located on the graph of function $f(x)$ in the second quadrant (see figure).

A tangent to the graph of function $f(x)$ is drawn through Point B .

The tangent intersects the y -axis at Point C .

Let t denote the x -coordinate of Point B .

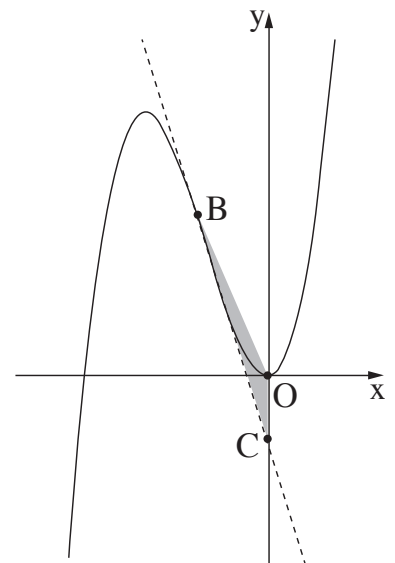
א. Express the equation of the tangent to the graph of function $f(x)$ through Point B in terms of t .

We know that Point C is located below the x -axis.

ב. What is the range of values of t ?

Point O is the origin.

ג. Find the maximum area of Triangle OBC .



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך