

Mathematics
5 Points — 1st Exam

מתמטיקה
5 יחידות לימוד — שאלון ראשון

Instructions for examinees

הוראות לנבחן

א. Duration of the exam: Three and a half hours

א. משך הבחינה: שלוש שעות וחצי.

ב. Exam structure and breakdown of points:

This exam has three parts and a total of eight questions.

Part One: Algebra and Probability

Part Two: Planar Geometry and Planar Trigonometry

Part Three: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה שלושה פרקים, ובהם שמונה שאלות.

פרק ראשון: אלגברה והסתברות

פרק שני: גאומטריה וטריגונומטריה במישור

פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,

של פונקציות שורש, של פונקציות רציונליות

ושל פונקציות טריגונומטריות

Answer five questions of your choice —

$5 \times 20 = 100$ points

עליך לענות על חמש שאלות לבחירתך — $20 \times 5 = 100$ נקודות.

ג. Material that may be used during the exam:

ג. חומר עזר מותר בשימוש:

(1) A non-graphing calculator. Do not use the

programming options of a programmable calculator.

Use of a graphing calculator or programming options
may lead to the disqualification of your exam.

(2) Formula sheets (attached).

(3) A Hebrew-foreign language/foreign language-Hebrew
dictionary.

(1) מחשבון לא גרפי.

אין להשתמש באפשרויות התכנות במחשבון שיש בו

אפשרות תכנות. שימוש במחשבון גרפי או באפשרויות

התכנות במחשבון עלול לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי-לועזי/לועזי-עברי.

ד. Special instructions:

ד. הוראות מיוחדות:

(1) Do not copy the question, but do indicate the number
of the question you are answering.

(1) אל תעתיק את השאלה; סמן את מספרה בלבד.

(2) Start each question on a new page.

(2) התחל כל שאלה בעמוד חדש.

Write the steps of the solution in the answer booklet,

even when the calculations are done with a calculator.

Explain all of your work, including calculations, in
detail and in a clear and organized fashion.

Lack of detail may lower your score or may lead to
the disqualification of your exam.

רשום במחברת את שלבי הפתרון, גם כאשר

החישובים מתבצעים בעזרת מחשבון.

הסבר את כל פעולותיך, כולל חישובים,

בפירוט ובצורה ברורה ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

Write only in the answer booklet. Write "טייטה" at the top of each draft
page. If you write draft material on paper outside the answer booklet,
your exam may be disqualified.

כתוב במחברת הבחינה בלבד. רשום "טייטה" בראש כל עמוד המשמש טיוטה.
כתיבת טיוטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Note: Explain all of your work, including calculations, clearly and in detail.

Lack of detail may lower your score or lead to the disqualification of your exam.

Answer five of the questions 1-8 (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Algebra and Probability

1. Yoav and Danny set off at the same time on a bike ride. They cycled along a straight route that started at Point A and ended at Point B .

Each cycled at a constant speed the whole time.

Yoav arrived at Point B and immediately returned to Point A along the same route.

When Yoav reached the midpoint of the route AB on his way back from B to A, Danny arrived at Point B .

- א. What is the ratio between Yoav's speed and Danny's speed? Explain.

40 minutes after they set off, when Yoav was on his way back from B to A , Yoav and Danny met.

- ב. Express the length of the route AB in terms of Danny's speed.

30 minutes after they set off, Yoav had still not arrived at Point B , and Danny's distance from Point A was 5 km greater than Yoav's distance from Point B .

- ג. Find the length of the route AB .

- ד. How much time passed from the moment Yoav and Danny set off from Point A until the distance between them was 2 km?

Find two of the three options.

2. A geometric sequence a_n satisfies the rule $3a_{n+2} + 5a_{n+1} - 2a_n = 0$ for any natural n .
Given: $a_1 \neq 0$.

א. Find the two possible values for the common ratio of the sequence a_n .

Let $b_1, b_2, b_3, \dots$ denote the terms of the **not convergent** [לא מתכנסת] sequence which satisfies the rule.

Let $c_1, c_2, c_3, \dots$ denote the terms of the **convergent** [מתכנסת] sequence which satisfies the rule.

ב. Explain why the sequence $b_1c_1, b_2c_2, b_3c_3, \dots$ is a convergent geometric sequence.

Given: $b_1c_1 + b_2c_2 + b_3c_3 + \dots = 15$

$$b_1 = c_1 = m$$

ג. Find m (list both possibilities).

Answer Item א for the smaller of the two possibilities for m that you found in Item ג.

ד. Given: $b_1 + b_2 + b_3 + \dots + b_k = 1,705$

Find k .

3. A barrel contains balls of three colors only: **red, yellow, blue**.

Given:

The probability of taking out a red ball is $\frac{5}{8}$.

The number of yellow balls is 3 times greater than the number of blue balls.

$\frac{4}{5}$ of the red balls in the barrel and $\frac{8}{9}$ of the yellow balls in the barrel are **rough**, and all the rest of the balls in the barrel are **smooth**.

A ball was randomly taken out of the barrel and then placed back in the barrel. This action (randomly taking out and then replacing) was repeated 8 times.

א. What is the probability that exactly 3 of the balls taken out are **rough**?

Answer Item ב for a barrel that contains 32 balls.

ב. 2 balls were randomly taken out of the barrel one after another (without replacing them).

(1) What is the probability that the two balls that were taken out are different colors?

(2) We know that the two balls that were taken out are different colors. What is the probability that the first ball that was taken out was red?

Answer Item ג for a barrel that contains n balls.

Given: $50 < n < 100$.

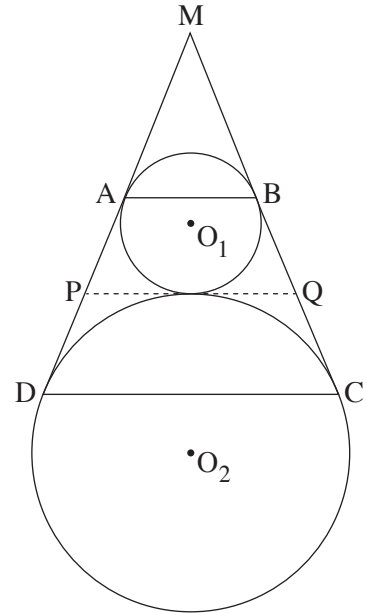
ג. Find n (both possibilities).

Part Two — Planar Geometry and Planar Trigonometry

4. The figure on the right describes two circles that are externally tangent to each other. Points O_1 and O_2 are the centers of these circles, and their radii are R_1 and R_2 , respectively.

Two lines that extend from Point M (which is located outside the circles) are tangent to the circle O_1 at Points A and B, and to the circle O_2 at Points D and C, as shown in the figure.

The tangent at the two circles' common point intersects the lines MD and MC at Points P and Q, respectively.



- א. Prove that Quadrilateral ABCD is an isosceles trapezoid [טרפז שווה שוקיים].
- ב. Prove that PQ is equal to the leg of Trapezoid ABCD.
- ג. Prove that $\angle O_1 Q O_2 = 90^\circ$.

Given: $R_1 = 4$, $R_2 = 9$

- ד. Find PQ.

5. The figure below describes an acute triangle ABC inscribed in a circle whose radius is R . The tangent to the circle at Point C intersects the extension of Segment AB at Point D. Given: The radius of the circle that circumscribes Triangle ACD is $2R$.

Let $\angle BAC = \alpha$.

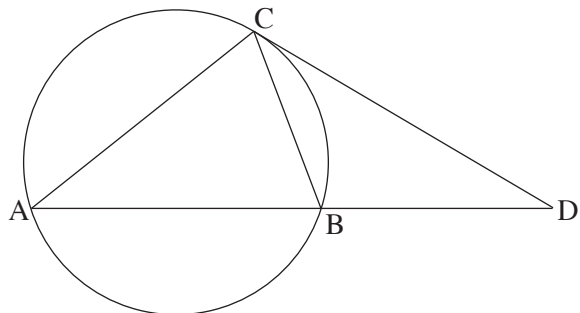
- א. Express BD in terms of R and α .

Given: $\frac{CD}{BD} = \frac{3}{2}$.

- ב. Find α .

Given: The area of Triangle CBD is 27.

- ג. Find R .



Part Three—Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Consider the given function $f(x) = \cos^3(x) \cdot \sin(x)$ on the domain $0 \leq x \leq \pi$.
- א. Find the coordinates of the extrema of function $f(x)$ and determine what type of extrema they are.
 - ב. Sketch a graph of function $f(x)$.

Consider the given function $g(x) = a \cdot f(x)$. $a > 0$ is a parameter.

- ג. Express in terms of a the equation of the tangent to the graph of function $g(x)$ at the point where $x = 0$.

The line you found in Item ג does not intersect the graph of function $g(x)$ at any other point.

Given: The area of the region bounded by the graph of function $g(x)$, the line you found in Item ג, and the line $x = \frac{\pi}{2}$, equals $\left(\frac{\pi^2}{2} - 1\right)$.

- ד. Find a .

7. Consider the given function $f(x) = \frac{x+a}{\sqrt{x}}$. a is a parameter.
- Find the domain [תחום ההגדרה] of the function $f(x)$.
 - For what values of the parameter a does function $f(x)$ not have extrema? Explain.
 - In the cases where function $f(x)$ does have an extremum, express the coordinates of the extremum in terms of a , and determine what type of extremum it is.
 - Sketch a separate graph of function $f(x)$ for each of the intervals i-iii of the parameter a listed below:
 - $a > 0$
 - $a < 0$
 - $a = 0$

Consider the given function $g(x) = f(x) - b$. b is a parameter.

Given: The graph of function $g(x)$ intersects the x-axis at two points.

- Find the possible range of values for the parameter a . Explain.
 - Express the possible range of values for the parameter b in terms of a . Explain.
8. Consider the given function $f(x) = x \cdot \sqrt{a - x^2}$. $a > 0$ is a parameter.
- Express the domain [תחום ההגדרה] of function $f(x)$ in terms of a .
 - Prove that the function $f(x)$ is odd [אי-זוגית].
 - The figure on the right describes part of the graph of function $f(x)$.

Copy the figure to your notebook and complete it so that it describes the entire graph of function $f(x)$.

A perpendicular is dropped to the x-axis through Point A, which is located in the first quadrant and on the graph of function $f(x)$.

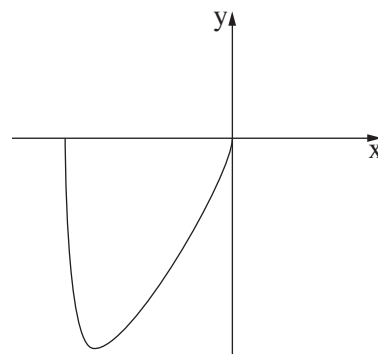
The perpendicular intersects the x-axis at Point B.

A line passing through Point A and the origin, O, intersects the graph of function $f(x)$ at another point, C.

A perpendicular is dropped to the x-axis through Point C. The perpendicular intersects the x-axis at Point D.

Given: The maximum sum of the areas of the triangles AOB and COD is $4\sqrt{2}$.

- Find a .



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל.
אין להעתיק או לפרסם אלא ברשות משרד החינוך.