

Mathematics
5 Points — 1st Exam

Instructions for examinees

א. Duration of the exam: Three and a half hours

ב. Exam structure and breakdown of points:

This exam has three parts and a total of eight questions.

Part One: Algebra and Probability

Part Two: Planar Geometry and Planar Trigonometry

Part Three: Differential and Integral Calculus of

Polynomials, Root Functions, Rational

Functions, and Trigonometric Functions

Answer five questions of your choice —

$$5 \times 20 = 100 \text{ points}$$

ג. Material that may be used during the exam:

(1) A non-graphing calculator.

Do not use the programming options

of a programmable calculator.

Use of a graphing calculator or programming options

may lead to the disqualification of your exam.

(2) Formula sheets (attached).

(3) A Hebrew-foreign language/foreign language-Hebrew dictionary.

ד. Special instructions:

(1) Do not copy the question, but do indicate the number of the question you are answering.

(2) Start each question on a new page.

Write the steps of the solution in the answer booklet,

even when the calculations are done with a calculator.

Explain all of your work, including calculations, in detail and in a clear and organized fashion.

Lack of detail may lower your score or may lead to the disqualification of your exam.

מתמטיקה

5 יחידות לימוד — שאלון ראשון

הוראות לנבחן

א. משך הבחינה: שלוש שעות וחצי.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה שלושה פרקים, ובהם שמונה שאלות.

פרק ראשון: אלגברה והסתברות

פרק שני: גאומטריה וטריגונומטריה במישור

פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,

של פונקציות שורש, של פונקציות רציונליות

ושל פונקציות טריגונומטריות

עליך לענות על חמש שאלות לבחירתך — $20 \times 5 = 100$ נקודות.

ג. חומר עזר מותר בשימוש:

(1) מחשבון לא גרפי.

אין להשתמש באפשרויות התכנות במחשבון

שיש בו אפשרות תכנות.

שימוש במחשבון גרפי או באפשרויות התכנות

במחשבון עלול לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי-לועזי/לועזי-עברי.

ד. הוראות מיוחדות:

(1) אל תעתיק את השאלה; סמן את מספרה בלבד.

(2) התחל כל שאלה בעמוד חדש.

רשום במחברת את שלבי הפתרון, גם כאשר

החישובים מתבצעים בעזרת מחשבון.

הסבר את כל פעולותיך, כולל חישובים,

בפירוט ובצורה ברורה ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או

לפסילת הבחינה.

Write only in the answer booklet. Write "טייטה" at the top of each draft page. If you write draft material on paper outside the answer booklet, your exam may be disqualified.

כתוב במחברת הבחינה בלבד. רשום "טייטה" בראש כל עמוד המשמש טייטה. כתיבת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

Good Luck!

בהצלחה!

Questions

Note: Explain all of your work, including calculations, clearly and in detail.

Lack of detail may lower your score or lead to the disqualification of your exam.

Answer five of the questions 1-8 (each question – 20 points).

Note: If you answer more than five questions, only the first five answers in your answer booklet will be graded.

Part One — Algebra and Probability

1. Two couriers, Ayal and Barak, set out toward each other at 8:00 am to deliver a package.

Ayal left from City A and Barak left from City B.

After Ayal had covered $\frac{1}{6}$ of the way to City B, he noticed that he had forgotten the package in City A. He returned to City A, collected the package, and immediately left for City B again. Ayal always traveled at a constant speed.

Barak also traveled at a constant speed, but his speed was 20% faster than Ayal's speed.

Barak and Ayal met at a point 75 km from City A.

- א. Find the length of the route between the two cities.

Ayal and Barak traveled on highways with speed limits of 50 to 110 km/h. Both Ayal and Barak traveled within the speed limit.

- ב. (1) Is it possible that Ayal and Barak met at 9:40 am? Explain.
(2) Is it possible that Ayal and Barak met at 10:00 am? Explain.

2. a_n is an infinite geometric sequence with a common ratio q .

Given: $0 < a_1$, $0 < q < 1$.

b_n is an increasing infinite geometric sequence with a common ratio r .

Given: $b_1 = a_6$.

The sequence c_n is defined as: $c_n = \frac{a_{n+5}}{b_n}$.

א. Explain why all the terms in all three sequences a_n , b_n , and c_n are positive.

ב. Prove that c_n is a geometric sequence, and find c_1 .

ג. (1) Explain why the common ratio of the sequence c_n is larger than 0 and smaller than 1.

(2) Given: The sum of the sequence c_n is $\frac{6}{5}$, $\frac{b_2}{a_8} = 18$.

Find q and r .

3. The probability of a child born in the Levi family having curly hair is x .

The probability of a child born in the Levi family having brown eyes is $2x$.

The probability of a child of born in the Levi family having brown eyes, if we know that the child's hair is curly, is smaller by a factor of 1.5 than the probability of this child's hair not being curly, if we know that the child's eyes are brown.

Yonatan is one of the Levi children.

א. (1) Show that the probability that Yonatan's eyes are brown and his hair is curly is $\frac{1}{2}x$.

(2) Find the probability of Yonatan's hair being curly if we know that his eyes are brown.

ב. (1) Express the probability that Yonatan's hair is not curly and his eyes are not brown in terms of x .

(2) Given: $x = 0.2$.

Exactly four children were born in the Levi family.

What is the probability that at least three of the four Levi children have curly hair and brown eyes?

Part Two — Planar Geometry and Planar Trigonometry

4. Line AD is tangent to a circle at Point A .

Point B is located on the circle such that Segment BD passes through the center of the circle, O , and intersects the circle at another point, E .

Point C is located on the circle such that $BC \parallel AD$.

Lines BD and AC intersect at Point M (see figure).

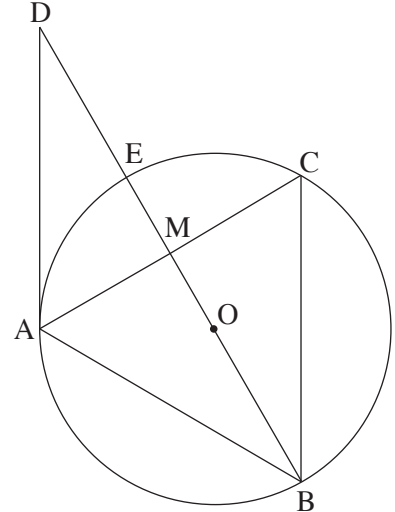
- א. Prove that $AB = AC$.

Given: AE bisects the angle MAD .

- ב. Prove that $BM \perp AC$.

- ג. Prove that the length of Segment AE is equal to the radius of the circle.

- ד. Prove that ABCD is a rhombus.



5. ABC is an obtuse triangle ($\angle BAC > 90^\circ$) .

Given: $AB + AC = 4a$ (a is a parameter),

$$AB : AC = 3 : 5,$$

the area of triangle ABC is $\frac{15\sqrt{3}}{16}a^2$.

- א. (1) Calculate the size of Angle BAC .

- (2) Calculate the size of the Angles ABC and ACB .

In the circle that circumscribes Triangle ABC , a regular pentagon can be inscribed. The area of the inscribed pentagon is 100 .

- ב. Calculate the value of a .

Part Three—Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

6. Given: the function $f(x) = 6x(x^3 - 1)^3$, which is defined for any x .

Answer Items ג-א. If necessary, leave two decimal places in your answer.

- א. (1) What are the coordinates of the intersection points of the graph of the function $f(x)$ with the axes?
- (2) Find the coordinates of the extrema [נקודות הקיצון] of the function $f(x)$, and determine what type of extrema they are (if there are any).
- (3) Sketch a graph of the function $f(x)$.
- (4) For which values of k is the line $y = k$ tangent to the graph of function $f(x)$?

- ב. Given: the equation $6x(x^3 - 1)^3 = m$. m is a parameter.

Using the graph of function $f(x)$, determine for which values of m the given equation has exactly two different positive solutions, and for which values of m it has one positive and one negative solution. Explain your answers.

- ג. Use the sketch to determine whether there is any $a > 0$ for which the integral $\int_0^a f(x) dx$ attains a minimum value.

If there is, what is the value of this a ? Explain your answer.

7. Given: The function $f(x) = 2 \sin^2 x - 1$ is defined for any x .

Answer Items ג-א for the interval $-\pi \leq x \leq \pi$.

- א. (1) Show that the function $f(x)$ is an even function.
- (2) Find the coordinates of the intersection points of the graph of the function $f(x)$ with the axes.
- (3) Find the coordinates of the extrema [נקודות קיצון] of the function $f(x)$ and determine what type of extrema they are.

Given: the function $g(x) = \frac{\cos 2x(1 - \sin x)}{\sin x - 1}$.

- ב. (1) What is the domain [תחום ההגדרה] of the function $g(x)$?
- (2) For which values of x does $f(x) = g(x)$? Explain.
- (3) Does the function $g(x)$ have vertical asymptotes? Explain.
- (4) Sketch a graph of the function $g(x)$.

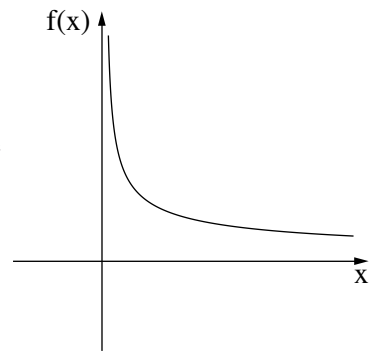
ג. Given: The domain of the function $h(x) = -f(x) + b$ (b is a parameter) is identical to the domain of the function $f(x)$.

Given: $\int_{-\pi}^0 h(x) dx = \frac{3\pi}{2}$. Find the value of the parameter b .

8. The figure on the right describes the graph of function $f(x) = \frac{4}{\sqrt{x}}$. The domain [תחום ההגדרה] of this function is $x > 0$.

Of all the points on the graph of function $f(x)$, Point A is the closest to the origin, O.

- א. (1) Find the coordinates of Point A.
- (2) Is Line AO perpendicular to the line that is tangent to the graph of function $f(x)$ at Point A? Explain.



Given: The function $g(x) = -f(-x)$ is defined on the domain $x < 0$.

Answer Item ב for the interval $-4 \leq x \leq -1$.

- ב. (1) Of all the points on the graph of the function $g(x)$ over the given interval, what are the coordinates of the point that is closest to the origin?
- (2) Find the coordinates of the point that is farthest from the origin of all the points on the graph of function $g(x)$ over the given interval.

Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך