

Mathematics
5 Points — 1st Exam

Instructions for examinees

א. Duration of the exam: Three and a half hours

ב. Exam structure and breakdown of points:

This exam has three parts.

Part One: Algebra and Probability

— 2×20 — 40 points

Part Two: Planar Geometry and Planar Trigonometry

— 1×20 — 20 points

Part Three: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions

— 2×20 — 40 points

Total — 100 points

ג. Material that may be used during the exam:

- (1) A non-graphing calculator. You may not use the programming options of a programmable calculator. Use of a graphing calculator or programming options may lead to the disqualification of your exam.
- (2) Formula sheets (attached).
- (3) A Hebrew-foreign language/foreign language-Hebrew dictionary.

ד. Special instructions:

- (1) Do not copy the question, but do indicate the number of the question you are answering.
- (2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator. Explain all of your work, including calculations, in detail and in a clear and organized fashion. Lack of detail may lower your score or may lead to the disqualification of your exam.

Write only in the answer booklet. Write "טייטה" at the top of each draft page. If you write draft material on paper outside the answer booklet, your exam may be disqualified.

Good Luck!

מתמטיקה

5 יחידות לימוד - שאלון ראשון

הוראות לנבחן

א. משך הבחינה: שלוש שעות וחצי.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה שלושה פרקים.

פרק ראשון: אלגברה והסתברות

— 2×20 — 40 נקודות

פרק שני: גאומטריה וטריגונומטריה במישור

— 1×20 — 20 נקודות

פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,
של פונקציות שורש, של פונקציות רציונליות
ושל פונקציות טריגונומטריות

— 2×20 — 40 נקודות

סה"כ — 100 נקודות

ג. חומר עזר מותר בשימוש:

(1) מחשבון לא גרפי.

אין להשתמש באפשרויות התכנות במחשבון הניתן לתכנות.

שימוש במחשבון גרפי או באפשרויות התכנות

במחשבון עלול לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי-לועזי/לועזי-עברי.

ד. הוראות מיוחדות:

(1) אל תעתיק את השאלה; סמן את מספרה בלבד.

(2) התחל כל שאלה בעמוד חדש.

רשום במחברת את שלבי הפתרון, גם כאשר

החישובים מתבצעים בעזרת מחשבון.

הסבר את כל פעולותיך, כולל חישובים,

בפירוט ובצורה ברורה ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

כתוב במחברת הבחינה בלבד. רשום "טייטה" בראש כל עמוד המשמש טייטה.
כתיבת טייטה בדפים שאינם במחברת הבחינה עלולה לגרום לפסילת הבחינה.

בהצלחה!

Questions

Note: Explain all of your work, including calculations, clearly and in detail.

Lack of detail may lower your score or lead to the disqualification of your exam.

Part One — Algebra and Probability (40 points)

Answer two of the questions 1-3 (each question – 20 points).

Note: If you answer more than two questions, only the first two answers in your answer booklet will be graded.

1. A bakery has two machines for manufacturing cakes: Machine I and Machine II.

Each machine manufactures cakes at its own constant rate.

On Sunday, both machines worked for the same amount of time.

On Sunday, Machine I manufactured 80 more cakes than Machine II.

On Monday, Machine II manufactured the same number of cakes that Machine I manufactured on Sunday, and Machine I manufactured the same number of cakes that Machine II manufactured on Sunday.

On Monday, the amount of time that Machine II worked was $\frac{25}{9}$ times longer than the amount of time that Machine I worked the same day.

- א. Calculate the total number of cakes the two machines manufactured on Sunday.

Let T_1 denote the time required for Machine I to manufacture one cake.

Let T_2 denote the time required for Machine II to manufacture one cake.

- ב. Calculate the ratio $\frac{T_1}{T_2}$. Explain.

- ג. (1) In a certain amount of time, Machine I manufactured exactly 47 cakes.

How many whole cakes did Machine II manufacture during this time? Explain.

- (2) If we know that the two machines worked for the same amount of time, and each of them manufactured a whole number [מספר שלם] of cakes, is it possible that in this amount of time, the two machines together manufactured 26 cakes? Explain.

2. a_n is an infinite geometric sequence whose common ratio is q . $|q| \neq 1$.

Given: $a_3 \cdot a_7 = 1$.

א. Calculate a_5 (find both possibilities).

Given: $a_5 > 0$.

א. (1) Express a_1 in terms of q .

(2) Is there a natural number n for which $a_n = \frac{1}{a_1}$? If so, find it. If not, explain.

(3) Is there a natural number n for which $a_n = \frac{1}{a_{13}}$? If so, find it. If not, explain.

א. (1) Express the first 7 terms of the sequence a_n in terms of q .

(2) Given: $a_1 \cdot a_2 \cdot \dots \cdot a_k = 1$ (k is a natural number).

Find the value of k , and explain why it is the only possible value of k .

3. Gali and Neta are playing a game in which the players may decide how many rounds to play. In each round, one of them wins and the other loses. The winner of the whole game is the player who wins more rounds than her opponent. If both of them win the same number of rounds, the result of the whole game is a draw.

Given: In each round, Neta's chance of winning is $\frac{1}{3}$.

א. On Sunday, Gali and Neta played a 4-round game.

(1) What is the probability that Neta won the whole game?

(2) What is the probability that the result of the whole game was a draw?

א. Gali and Neta played 4 rounds on Monday too. This time, they decided in advance that if the result after 4 rounds was a draw, they would play 3 more rounds in order to decide the final result, and the player who wins more rounds, wins the whole game.

What is the probability that Neta will win the whole game?

א. If we know that Neta won the whole game on exactly one of the two days, Sunday or Monday, what is the chance that she won the whole game on Monday?

Part Two — Planar Geometry and Planar Trigonometry (20 points)

Answer one of the questions 4-5.

Note: If you answer more than one question, only the first answer in your answer booklet will be graded.

4. EG is a chord [מיתר] in a circle whose center is M and whose radius is r . Tangents to the circle were drawn through Points E and G.

A line was drawn through the center of the circle, M, parallel to Chord EG and intersecting the tangents at Points K and L, as shown in the figure.

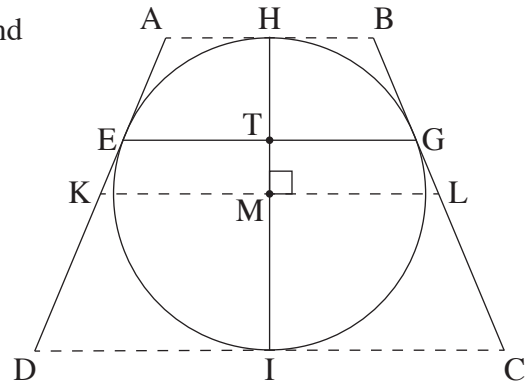
A perpendicular to KL was drawn through the center of the circle, M, intersecting Chord EG at Point T and intersecting the circle at Points H and I, as shown in the figure.

Let $TG = a$.

- ⌘. (1) Prove: $TG \cdot ML = MG^2$.
(2) Express the length of Segment KL in terms of a and r .

Tangents to the circle were drawn through Points H and I, forming an isosceles trapezoid ABCD that circumscribes the circle, as shown in the figure.

- Ⓐ. (1) Prove: $BC = KL$.
(2) Express the perimeter of Trapezoid ABCD in terms of a and r .
Ⓔ. Can the ratio between the perimeter of Trapezoid ABCD and the circumference of the circle be less than $\frac{4}{\pi}$? Explain.



5. ABCD is a rhombus. The length of its side is a .

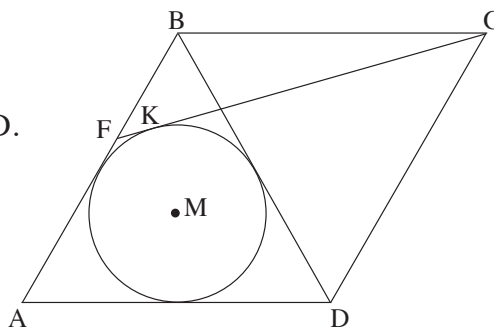
Given: $\angle BAD = 60^\circ$.

A circle whose center is M is inscribed in Triangle ABD.

A tangent to the circle was drawn from Vertex C.

The tangent's extension intersects Side AB at Point F.

Point K is the point of tangency (see figure).



א. Express the radius of the circle in terms of a .

ב. (1) Explain why Point M is located on Diagonal AC of the rhombus.

(2) Calculate the size of Angle ACF .

ג. Express the area of Triangle ACF in terms of a .

Part Three—Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

(40 points)

Answer two of the questions 6-8 (each question – 20 points).

Note: If you answer more than two questions, only the first two answers in your answer booklet will be graded.

6. Given: the family of functions $f(x) = \frac{\sqrt{x^2 + x - 2}}{2x - a}$. a is a parameter that satisfies $-4 < a < 2$.

- א. (1) Find the domain [תחום ההגדרה] of the function $f(x)$.
- (2) Explain why the function $f(x)$ has no asymptote parallel to the y-axis.
- (3) Find the equations of the asymptotes of the function $f(x)$ that are parallel to the x-axis.
- (4) What are the coordinates of the points of intersection of the graph of the function $f(x)$ with the axes?
- (5) Find the positive and negative intervals of the function $f(x)$.
- ב. (1) Express in terms of a the values of x for which $f'(x) = 0$ (if there are any).
- (2) Find the value of a for which $f'(x) \neq 0$ for all values of x in the domain.

Substitute $a = -1$ in the equation of the function $f(x)$ and answer Items ג and ד.

- ג. (1) What are the intervals on which the function $f(x)$ is increasing and decreasing (if there are such intervals)?
- (2) Sketch a graph of the function $f(x)$.

ד. Calculate $\int_3^4 \frac{1}{f(x)} dx$. You may leave a root sign in your answer.

7. Given: the function $f(x) = x^3 \sin x$ that is defined on the domain $-\pi \leq x \leq \pi$.

- א. (1) Determine whether the function $f(x)$ is even, odd, or neither even nor odd. Explain.
(2) Find the coordinates of the points of intersection of the graph of the function $f(x)$ with the x -axis on the given interval.
(3) Explain why the function $f(x)$ is non-negative on the given interval.
(4) Determine whether the derivative function, $f'(x)$, is even, odd, or neither even nor odd. Explain.

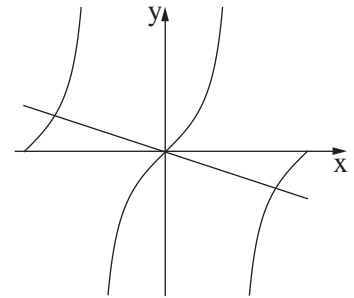
ב. (1) Show that the x -coordinates for which $f'(x) = 0$ satisfy the equation $\tan x = -\frac{1}{3}x$.

(2) The figure at the right shows the graphs of the functions

$$h(x) = -\frac{1}{3}x \quad \text{and} \quad g(x) = \tan x \quad \text{on the interval}$$

$$-\pi \leq x \leq \pi.$$

Using the figure, determine how many points on the interval $-\pi \leq x \leq \pi$ satisfy $f'(x) = 0$.

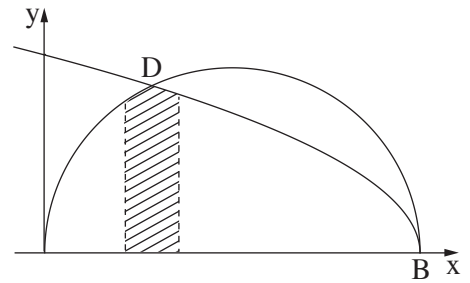


Given: The x -coordinate of one of the extrema [נקודות הקיצון] of the function $f(x)$ is approximately 2.46.

Answer Items א and ב for the interval $-\pi \leq x \leq \pi$.

- א. (1) What are the x -coordinates of all the extrema of the function $f(x)$ on the interval? Explain, and determine what kind of extrema they are.
(2) Sketch a graph of the function $f(x)$ on the interval.
ב. (1) Sketch a graph of the derivative function, $f'(x)$, on the interval.
(2) What is the minimum number of inflection points [נקודות פיתול] that the function $f(x)$ has on the interval? Explain.

8. The figure at the right shows the graphs of the functions $f(x) = \sqrt{-x^2 + 7x}$ and $g(x) = \sqrt{14 - 2x}$. The graph of the function $f(x)$ intersects the x -axis at the origin and at Point B, and it intersects the graph of the function $g(x)$ at Points B and D, as shown in the figure.



- א. (1) Find the domains [תחומי ההגדרה] of the functions $f(x)$ and $g(x)$.
(2) Find the x -coordinates of Points B and D.

a is a parameter that satisfies $1 \leq a \leq 2$.

The region bounded by the graphs of the functions $f(x)$ and $g(x)$, the perpendiculars $x = a$ and $x = a + 1$, and the x -axis, is rotated around the x -axis.

- ב. (1) Calculate the value of a for which the volume of the obtained solid of revolution [גוף הסיבוב] is maximal.
(2) Find the value of a for which the volume of the obtained solid of revolution is minimal.

If necessary, you may leave two decimal places in your answers.

Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך