

Mathematics
5 Points — 1st Exam

Instructions for examinees

א. Duration of the exam: Three and a half hours

ב. Exam structure and breakdown of points:

This exam has three parts.

Part One: Algebra and Probability

– $2 \times 20 = 40$ points

Part Two: Planar Geometry and Planar Trigonometry

– $1 \times 20 = 20$ points

Part Three: Differential and Integral Calculus of
Polynomials, Root Functions, Rational
Functions, and Trigonometric Functions

– $2 \times 20 = 40$ points

Total – 100 points

ג. Material that may be used during the exam:

- (1) A non-graphing calculator. You may not use the programming options of a programmable calculator. Use of a graphing calculator or programming options may lead to the disqualification of your exam.
- (2) Formula sheets (attached).
- (3) A Hebrew-foreign language/foreign language-Hebrew dictionary.

ד. Special instructions:

- (1) Do not copy the question, but do indicate the number of the question you are answering.
- (2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator. Explain all of your work, including calculations, in detail and in a clear and organized fashion. Lack of detail may lower your score or may lead to the disqualification of your exam.
- (3) For draft paper, use pages in the answer booklet. Use of other draft paper may lead to the disqualification of your exam.

Good Luck!

מתמטיקה

5 יחידות לימוד - שאלון ראשון

הוראות לנבחן

א. משך הבחינה: שלוש שעות וחצי.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה שלושה פרקים.

פרק ראשון: אלגברה והסתברות

– $20 \times 2 = 40$ נקודות

פרק שני: גאומטריה וטריגונומטריה במישור

– $20 \times 1 = 20$ נקודות

פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,
של פונקציות שורש, של פונקציות רציונליות
ושל פונקציות טריגונומטריות

– $20 \times 2 = 40$ נקודות

סה"כ – 100 נקודות

ג. חומר עזר מותר בשימוש:

- (1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות במחשבון הניתן לתכנות. שימוש במחשבון גרפי או באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.
- (2) דפי נוסחאות (מצורפים).
- (3) מילון עברי-לועזי/לועזי-עברי.

ד. הוראות מיוחדות:

- (1) אל תעתיק את השאלה; סמן את מספרה בלבד.
- (2) התחל כל שאלה בעמוד חדש. רשום במחברת את שלבי הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון. הסבר את כל פעולותיך, כולל חישובים, בפירוט ובצורה ברורה ומסודרת. חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.
- (3) לטייטה יש להשתמש במחברת הבחינה. שימוש בטייטה אחרת עלול לגרום לפסילת הבחינה.

בהצלחה!

Questions

Note: Explain all of your work, including calculations, clearly and in detail.

Lack of detail may lower your score or lead to the disqualification of your exam.

Part One — Algebra and Probability (40 points)

Answer two of the questions 1-3 (each question – 20 points).

Note: If you answer more than two questions, only the first two answers in your answer booklet will be graded.

1. A group of experienced lumberjacks planned to cut down 216 cubic meters of timber over a certain number of days (the workers' daily output was constant).

During the first three days the workers proceeded at the planned output. From the fourth day onward they worked faster and each day cut down 8 cubic meters of timber more than planned. They actually worked one day less than planned, and cut down a total of 232 cubic meters of timber.

- א. (1) According to the plan, how many cubic meters of timber were the workers supposed to cut down per day?
(2) How many days did the workers actually work?
- ב. On which day, counting from the day they started working, did the workers finish cutting down $\frac{2}{3}$ of the planned quantity?

Each of the experienced workers was then assigned an apprentice and a new group was formed with a total of $2m$ workers (m experienced workers and m apprentices).

The experienced workers produced the planned daily output. All the experienced workers have the same daily output. The daily output of an apprentice is smaller by 1 cubic meter than the daily output of an experienced worker. The new group worked for 8 days.

- ג. (1) Express the daily output of a single experienced worker and of a single apprentice, in terms of m .
(2) How many lumberjacks are there in total in the new group, if it is known that they cut down a total of 336 cubic meters of timber?

2. Given: an arithmetic sequence $a_1, a_2, \dots, a_{2n+3}$ with $2n+3$ terms (n is a natural number). The sum of the sequence is 43 times greater than its middle term. The middle term is not 0.

- א. (1) Show that the sum of the sequence equals $(2n+3) \cdot a_{n+2}$.
(2) Find the number of terms in the sequence.
- ב. For the given sequence, it is known that the sum of the terms in the odd positions is greater by 40 than the sum of the terms in the even positions.
(1) Find the middle term.
(2) Find the sum of the sequence.

Given: The common difference of the given sequence is $(-a_1)$.

- ג. Determine whether the sequence is increasing or decreasing.

All the terms of the given sequence are used to construct a new sequence by adding up every k adjacent terms (k is a natural number), as follows:

$$(a_1 + a_2 + \dots + a_k), (a_2 + a_3 + \dots + a_{k+1}), (a_3 + a_4 + \dots + a_{k+2}), \dots$$

- ד. Express in terms of k the number of terms in the new sequence.
3. The 12th grade students in a high school take the mock exam (מתכונת) in civics and then take the final exam (בגרות) in civics.

Given: Both in 2017 and in 2018 the number of students who passed the mock exam and failed the final exam was the same as the number of students who failed the mock exam and passed the final exam.

- א. In 2017, 250 students took the mock exam and then took the final exam. It is known that if a student passed the mock exam, the probability of the same student having passed the final exam is 0.9.

20% of the total number of students who took the exams that year, failed the final exam.

- (1) How many students passed both the mock exam and the final exam?
(2) It is known that a certain student failed the mock exam. What is the probability that this student passed the final exam?
(3) We randomly select (with replacement) two students who have failed the final exam. What is the probability that both of them also failed the mock exam?
- ב. Given: In 2018 the event "pass the mock exam" was independent of the event "pass the final exam", and the probability of a student passing the final exam in 2018 was a ($0 < a < 1$).
Express in terms of a the probability that a student passed the mock exam and failed the final exam that year.

Part Two — Planar Geometry and Planar Trigonometry

(20 points)

Answer one of the questions 4-5.

Note: If you answer more than one question, only the first answer in your answer booklet will be graded.

4. Triangle BCF is inscribed in a circle whose center is O and radius is R . BF is a diameter of the circle. Two tangents to the circle are drawn from Point A — one is tangent to the circle at Point B , and the other intersects the extension of Side CF at Point D , as shown in the figure on the right.

Given: $AD \perp CD$.

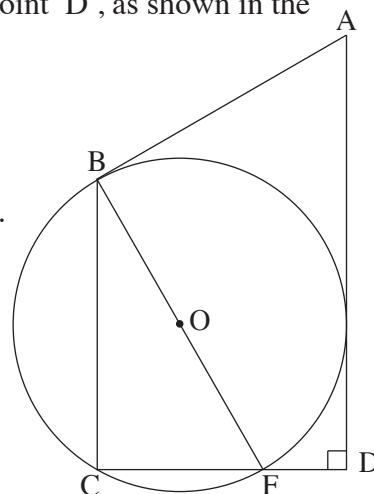
א. Prove that: $\angle BFC = \angle BAD$.

Given: K is a point on Side BC , such that FK bisects $\angle BFC$.

ב. Prove that: $KC = \frac{CF \cdot BO}{AB}$.

ג. Prove that: $KB \cdot AB = 2R^2$.

ד. Explain why the area of $\triangle BFK$ is greater than the area of $\triangle KFC$.



5. The triangle ABC is inscribed in a circle whose radius is R . The points D and E are located on the sides AB and AC , respectively, and the point F is located on the arc BC such that Quadrilateral $ADFE$ is a rhombus (see figure).

Given: $\angle BAC = \alpha$, $\angle ABC = \beta$.

א. (1) Express $\angle ABF$ in terms of α and β .

(2) Express the length of the diagonal AF in terms of R , α , and β .

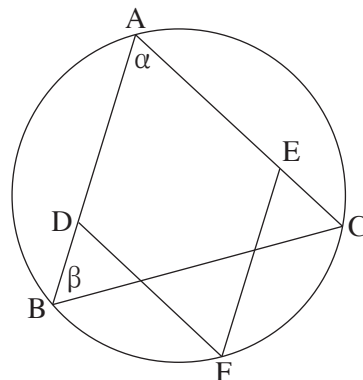
ב. Express the length of the side of the rhombus in terms of R , α , and β .

Given: AF is a diameter of the circle.

ג. Show that the area of the rhombus is $2R^2 \tan \frac{\alpha}{2}$.

Given: The radius of the circle inscribed in the rhombus $ADFE$ is $\frac{3}{5}R$.

ד. Calculate β .



Part Three—Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions, and Trigonometric Functions

(40 points)

Answer two of the questions 6-8 (each question – 20 points).

Note: If you answer more than two questions, only the first two answers in your answer booklet will be graded.

6. Given: The function $g''(x) = -\frac{18}{x^4} + \frac{18}{(x-4)^4}$ is the second derivative function of the function $g(x)$.

The functions $g(x)$, $g'(x)$, $g''(x)$ are defined over the same domain.

Given: The equation of the tangent to function $g(x)$ at its inflection point [נקודת פיתול] is $y = \frac{3}{2}x - 3$.

- א. (1) Find the function $g(x)$.
- (2) What is the domain [תחום ההגדרה] of the function $g(x)$?
- (3) Find the intervals on which the function $g(x)$ is increasing and decreasing.
- (4) Sketch the graph of the function $g(x)$.

Let: $h(x) = |g(x)|$.

- ב. On the same set of axes on which you sketched the graph of function $g(x)$, add a sketch of the graph of the function $h(x)$ (use a dashed line).

- ג. Given: $\int_a^2 g(x) dx = t$, $0 < a < 2$, t is a parameter.

Express $\int_a^2 (h(x) - g(x)) dx$ in terms of t .

7. Given: the function $f(x) = 2 \sin x + \cos 2x - 1$ over the interval $-\frac{\pi}{2} \leq x \leq \frac{3}{2}\pi$.

- א. (1) Find the coordinates of the points at which the graph of the function $f(x)$ intersects the axes.
- (2) Find the coordinates of the extrema [נקודות הקיצון] of the function $f(x)$ and determine what type of extrema they are.
- (3) Sketch the graph of the function $f(x)$.

The graph of the function $f(x)$ is shifted left by $\frac{\pi}{2}$, producing the function $g(x)$ whose domain is $-\pi \leq x \leq \pi$.

- א. (1) Express the function $g(x)$ in terms of the function $f(x)$.
- (2) Sketch the graph of the function $g(x)$.
- (3) Prove that $g(x)$ is an even function.

Given below are 3 expressions, I-III:

I: $\int_0^{\frac{\pi}{2}} f(x + \pi) dx$, II: $\int_{-\frac{\pi}{2}}^0 f\left(x + \frac{\pi}{2}\right) dx$, III: $\int_0^{\frac{\pi}{2}} f\left(x - \frac{\pi}{2}\right) dx$:

- א. Specify which of the expressions I-III equals $\int_0^{\frac{\pi}{2}} f(x) dx$.
- Explain your answer. There is no need for calculations.

8. Given: In Triangle ABC: $AB = 30$, $AC = 20$.

$\angle CAB = \alpha$. α is a constant.

Point D is located on Side AB and Point E is located on Side AC (see figure below).

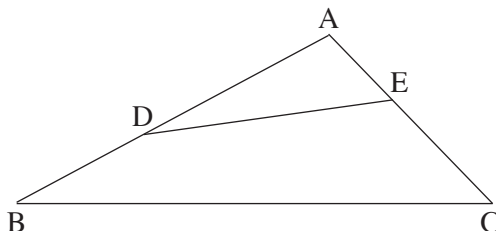
Given: The area of Triangle ADE formed in this way is a quarter of the area of Triangle ABC.

Let x be the length of the segment AD.

א. Express the length of the segment AE in terms of x .

ב. (1) Express the minimal length of the segment DE in terms of α .

(2) Derive from Sub-Item ב(1) the value of x for which the ratio $\frac{DE}{BC}$ is the smallest [מינימלי]. Explain.



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך