

Mathematics

5 Points — 1st Exam

Instructions for examinees

א. Duration of the exam: Three and a half hours

ב. Exam structure and breakdown of points:

This exam has three parts.

Part One: Algebra and Probability

— 2×20 — 40 points

Part Two: Planar Geometry and Planar Trigonometry

— 1×20 — 20 points

Part Three: Differential and Integral Calculus of

Polynomials, Root Functions, Rational

Functions, and Trigonometric Functions

— 2×20 — 40 points

Total — 100 points

ג. Material that may be used during the exam:

- (1) A non-graphing calculator. You may not use the programming options of a programmable calculator. Use of a graphing calculator or programming options may lead to the disqualification of your exam.
- (2) Formula sheets (attached).
- (3) A Hebrew-foreign language/foreign language-Hebrew dictionary.

ד. Special instructions:

- (1) Do not copy the question, but do indicate the number of the question you are answering.
- (2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator. Explain all of your work, including calculations, in detail and in a clear and organized fashion. Lack of detail may lower your score or lead to the disqualification of your exam.
- (3) For draft paper, use pages in the answer booklet. Use of other draft paper may lead to the disqualification of your exam.

Good Luck!

מתמטיקה

5 יחידות לימוד — שאלון ראשון

הוראות לנבחן

א. משך הבחינה: שלוש שעות וחצי.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה שלושה פרקים.

פרק ראשון: אלגברה והסתברות

— 2×20 — 40 נקודות

פרק שני: גאומטריה וטריגונומטריה במישור

— 1×20 — 20 נקודות

פרק שלישי: חשבון דיפרנציאלי ואינטגרלי של פולינומים,

של פונקציות שורש, של פונקציות רציונליות

ושל פונקציות טריגונומטריות

— 2×20 — 40 נקודות

סה"כ — 100 נקודות

ג. חומר עזר מותר בשימוש:

(1) מחשבון לא גרפי. אין להשתמש באפשרויות התכנות

במחשבון הניתן לתכנות.

שימוש במחשבון גרפי או באפשרויות התכנות במחשבון

עלול לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי-לועזי/לועזי-עברי.

ד. הוראות מיוחדות:

(1) אל תעתיק את השאלה; סמן את מספרה בלבד.

(2) התחל כל שאלה בעמוד חדש. רשום במחברת את שלבי

הפתרון, גם כאשר החישובים מתבצעים בעזרת מחשבון.

הסבר את כל פעולותיך, כולל חישובים, בפירוט ובצורה

ברורה ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת

הבחינה.

(3) לטייטה יש להשתמש במחברת הבחינה.

שימוש בטייטה אחרת עלול לגרום לפסילת הבחינה.

בהצלחה!

Questions

Note: Explain all of your work, including calculations, clearly and in detail.

Lack of detail may lower your score or lead to the disqualification of your exam.

Part One — Algebra and Probability (40 points)

Answer two of the questions 1-3 (each question – 20 points).

Note: If you answer more than two questions, only the first two answers in your answer booklet will be graded.

1. Two cyclists, Amir and Moshe, left at 6:00 AM, riding toward each other.

Amir cycled at a constant speed from Town α to Town β , and Moshe cycled at a constant speed from Town β to Town α .

Amir and Moshe passed each other and each continued on toward his destination.

Amir arrived at Town β two hours after he had passed Moshe, and

Moshe arrived at Town α eight hours after he had passed Amir.

α . At what time did Amir and Moshe pass each other?

Let V denote Amir's traveling speed.

At the exact moment that Amir and Moshe passed each other, Yasmin, who was riding a motorbike, left Town α in the direction of Town β , traveling at a constant speed.

Given: Yasmin arrived at Town β after Amir, but before Moshe arrived at Town α .

- β . (1) Express the distance between Town α and Town β in terms of V .
(2) Express the range of possible speeds for Yasmin in terms of V .

2. a_n is a convergent infinite geometric sequence whose sum is negative.

a_1 is the first term in the sequence, and q is the common ratio.

α . Read the four statements (I-IV), below. Only one statement must be true.

Write down the number of that statement and explain.

I) $q < 0$

II) $a_1 < 0$ and $q < 0$

III) $a_1 < 0$

IV) $a_1 > 0$ or $q < 0$

(Note: This question continues on the next page.)

Let T denote the sum of the terms in the odd positions in the sequence a_n ,
and let R denote the sum of the terms in the even positions in the sequence a_n .
 p is a parameter.

Given: $T + p \cdot R = 0$.

2. Express p in terms of q .

b_n is a geometric sequence whose common ratio is p .

3. Is b_n a convergent sequence? Explain.

7. Given: p is negative. Show that for any natural n , $a_{n+1} > a_n$
(in other words, show that the sequence a_n is an increasing sequence).

3. In a large city, all the high school students took an exam.

37% of the students who took the exam got help from their friends when they prepared for the exam.

$\frac{35}{37}$ of them passed the exam.

The number of students who did not get help from their friends and did not pass the exam is 5 times smaller than the number of students who did get help from their friends and did pass the exam.

8. A student who took the exam was chosen at random, and it turned out that he did not pass the exam. What is the probability that he got help from his friends?

2. Yael and Hadas took the exam. We know that Yael got help from her friends when she prepared for the exam, and that Hadas did not get help from her friends when she prepared for the exam. Is the probability that Yael passed the exam higher than the probability that Hadas passed the exam? Explain.

3. 6 students who took the exam were selected at random.

What is the probability that exactly one third of them did not get help from their friends and did pass the exam?

7. A student who took the exam was selected at random. What is the probability that this student satisfies at least one of the two statements, I-II:

I) This student got help from his friends.

II) This student did not pass the exam.

Part Two — Planar Geometry and Planar Trigonometry (20 points)

Answer one of the questions 4-5.

Note: If you answer more than one question, only the first answer in your answer booklet will be graded.

4. ABCD is a rhombus. E and F are the midpoints of Sides AB and BC, respectively.

Point K is the point of intersection of the diagonals of the rhombus.

A perpendicular to AB is drawn from Point E, intersecting the extension of Diagonal BD at Point G (see figure).

- א. Prove: Point G is the center of the circle that circumscribes Triangle ABC.

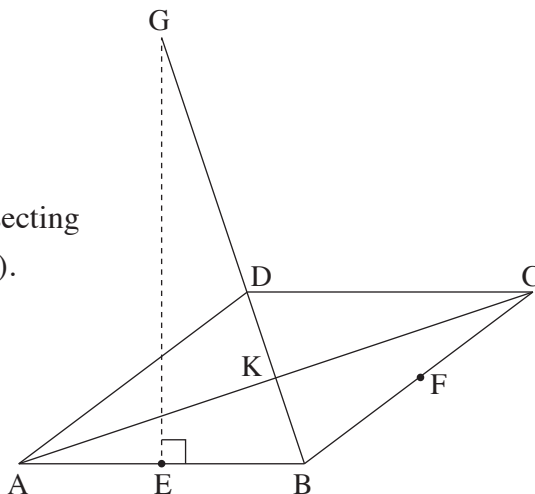
Segment GF intersects Diagonal AC at Point M, which is the center of the circle that circumscribes Triangle BDC.

- ב. Prove that Triangles MFC, BKC, and BFG are similar to each other.

Let R denote the radius of the circle circumscribing Triangle ABC, and let r denote the radius of the circle circumscribing Triangle BDC.

- ג. (1) Prove that $\frac{MC}{GB} = \frac{MF}{CF}$ and that $\frac{MF}{CF} = \frac{BK}{CK}$.

- (2) Show that the ratio between the diagonals of the rhombus is $\frac{r}{R}$.



5. ABC is a right triangle ($\angle ABC = 90^\circ$).

M is a point on the hypotenuse such that $AM : MC = \sqrt{3} : 4$.

Given: $\angle ABM = 30^\circ$, $BM = 8$.

- א. (1) Let $MC = 4x$; calculate the angles of Triangle ABC.

- (2) Calculate the radii of the circles that circumscribe Triangles ABM and CMB.

- ב. Let O_1 and O_2 denote the centers of the circles that circumscribe Triangles ABM and CBM, respectively.

- (1) Explain why the quadrilateral BO_1MO_2 is a kite [דלתון].

- (2) Calculate the length of Segment O_1O_2 .

**Part Three—Differential and Integral Calculus of Polynomials,
Root Functions, Rational Functions, and
Trigonometric Functions (40 points)**

Answer two of the questions 6-8 (each question – 20 points).

Note: If you answer more than two questions, only the first two answers in your answer booklet will be graded.

6. Given: The function $f(x) = \frac{ax - 1}{\sqrt{ax^2 - 2x + 1}}$. a is a parameter.

Given: The function $f(x)$ is defined for all x .

- א. Prove that $a > 1$.

Answer Item א. If necessary, express in terms of a .

- ב. (1) Find the coordinates of the points of intersection of the graph of function $f(x)$ with the axes.
(2) Write the equations of the asymptotes of function $f(x)$ that are parallel to the x -axis.
(3) Find the intervals on which the function $f(x)$ is increasing and decreasing (if there are any such intervals).
(4) Sketch the graph of the function $f(x)$.

Given: $a = 3$.

- ג. Calculate the area of the region bounded by the graph of the function $f(x)$, the x -axis, and the lines $x = \frac{2}{3}$ and $x = 2$.

- ד. $g(x)$ is a continuous function that is defined for all x .

Let S denote the area of the region bounded by the graph of the function $f(x)$, the x -axis, and the lines $x = \frac{1}{3}$ and $x = b$ ($b > \frac{1}{3}$).

Given: The area of the region bounded by the graph of the function $f(x)$, the graph of the function $g(x)$, and the lines $x = \frac{1}{3}$ and $x = b$ equals $2S$ for every b .

Express $g(x)$ in terms of $f(x)$ on the interval $x > \frac{1}{3}$ (list both possibilities). You do not need to prove your answer.

7. $f(x)$ is a differentiable function [פונקציה גזירה] that is defined for all x , such that $f(x) \neq 0$ for all x .

- א. Prove that if the function $f(x)$ increases on a certain interval, then the function $\frac{1}{f(x)}$ decreases on that interval; and if the function $f(x)$ decreases on a certain interval, then the function $\frac{1}{f(x)}$ increases on that interval.

Given: The function $g(x) = \sin^2 x + \cos x + 2$, which is defined for all x .

- א. Is there a value of x for which $g(x) = 0$? Explain.
- ג. (1) Is the function $g(x)$ an even function? Explain.
(2) Show that $g(x) = g(x + 2\pi)$ for all x .
(3) Find the coordinates of the extrema [נקודות קיצון] of the function $g(x)$ in the interval $0 \leq x \leq \pi$, and determine what kind of extrema they are.
(4) Sketch the graph of the function $g(x)$ on the interval $-\pi \leq x \leq 3\pi$.

Given: The function $h(x) = \frac{1}{\sin^2 x + \cos x + 2}$.

Answer Item 7. You may use your answers to the previous items.

7. (1) What is the domain [תחום ההגדרה] of the function $h(x)$? Explain.
(2) Sketch the graph of the function $h(x)$ on the interval $-\pi \leq x \leq \pi$ on the same set of axes you used to sketch the graph of the function $g(x)$.

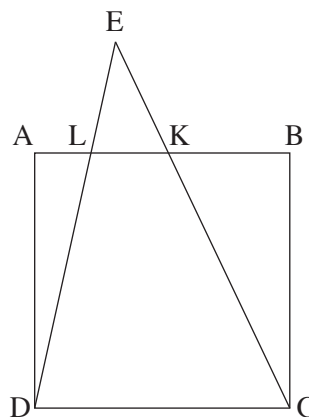
8. ABCD is a square, each of whose sides is 6 cm long.

K and L are points on Side AB.

Given: Lines CK and DL intersect at Point E, which is located outside Square ABCD (see figure).

Let $LK = x$.

- א. Express the height of Triangle KLE in terms of x .
א. For what value of x is the sum of the areas of Triangles BCK, ADL, and KLE minimal (smallest)? Explain.
You may leave a root sign in your answer.



Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך