

Mathematics

5 Points — First Questionnaire

Instructions for examinees

א. Duration of the exam: Three and a half hours

ב. Exam structure and breakdown of points:

This exam has three parts.

Part One: Algebra and Probability

$$- 2 \times 16 \frac{2}{3} - 33 \frac{1}{3} \text{ points}$$

Part Two: Planar Geometry and Planar Trigonometry

$$- 2 \times 16 \frac{2}{3} - 33 \frac{1}{3} \text{ points}$$

Part Three: Differential and Integral Calculus

$$- 2 \times 16 \frac{2}{3} - 33 \frac{1}{3} \text{ points}$$

Total — 100 points

ג. Material that may be used during the exam:

(1) A non-graphing calculator. You may not use the programming options of a programmable calculator. Use of a graphing calculator or programming options may lead to the disqualification of the exam.

(2) Formula sheets (attached).

(3) A Hebrew-foreign language/
foreign language-Hebrew dictionary.

ד. Special instructions:

(1) Do not copy the question, but do indicate the number of the question you are answering.

(2) Start each question on a new page. Write the steps of the solution in the answer booklet, even when the calculations are done with a calculator. Explain all of your work, including calculations, in detail and in a clear and organized fashion. Lack of detail may lower your score or lead to the disqualification of the exam.

(3) For draft paper, use pages in the answer booklet or paper you receive from the proctors. Use of other draft paper may lead to the disqualification of the exam.

מתמטיקה

5 יחידות לימוד - שאלון ראשון

הוראות לנבחן

א. משך הבחינה: שלוש שעות וחצי.

ב. מבנה השאלון ומפתח ההערכה:

בשאלון זה שלושה פרקים.

פרק ראשון: אלגברה והסתברות

$$33 \frac{1}{3} - 16 \frac{2}{3} \times 2 \text{ נקודות}$$

פרק שני: גאומטריה וטריגונומטריה במישור

$$33 \frac{1}{3} - 16 \frac{2}{3} \times 2 \text{ נקודות}$$

פרק שלישי: חשבון דיפרנציאלי ואינטגרלי

$$33 \frac{1}{3} - 16 \frac{2}{3} \times 2 \text{ נקודות}$$

סה"כ — 100 נקודות

ג. חומר עזר מותר בשימוש:

(1) מחשבון לא גרפי. אין להשתמש

באפשרויות התכנות במחשבון הניתן

לתכנות. שימוש במחשבון גרפי או

באפשרויות התכנות במחשבון עלול לגרום לפסילת הבחינה.

(2) דפי נוסחאות (מצורפים).

(3) מילון עברי-לועזי/לועזי-עברי.

ד. הוראות מיוחדות:

(1) אל תעתיק את השאלה; סמן את מספרה בלבד.

(2) התחל כל שאלה בעמוד חדש. רשום

במחברת את שלבי הפתרון, גם כאשר

החישובים מתבצעים בעזרת מחשבון.

הסבר את כל פעולותיך, כולל חישובים,

בפירוט ובצורה ברורה ומסודרת.

חוסר פירוט עלול לגרום לפגיעה בציון או לפסילת הבחינה.

(3) לטייטה יש להשתמש במחברת הבחינה או בדפים שקיבלת מהמשגיחים.

שימוש בטייטה אחרת עלול לגרום לפסילת הבחינה.

Good Luck!

/Continued on the next page/

בהצלחה!

/המשך מעבר לדף/

Questions

Note! Explain all of your work, including calculations, clearly and in detail. Lack of detail may lower your score or lead to the disqualification of your exam.

Part One — Algebra and Probability ($33\frac{1}{3}$ points)

Answer two of the questions 1-3 (each question – $16\frac{2}{3}$ points).

Note! If you answer more than two questions, only the first two answers in your answer booklet will be graded.

1. Worker I and Worker II are employed by a factory that produces replacement parts. The two workers work together to carry out a particular task.
 The pace at which Worker I normally works is different from the pace at which Worker II normally works.
 If each worker increases his pace by 50%, the difference between the time it would take the two workers to finish the job working together at their normal pace and the time it would take them to finish the job working together at their increased pace will equal $\frac{2}{15}$ of the time it would take for Worker I to finish the job if he were working alone at his normal pace.
 - א. Find the ratio of the time it takes Worker I to complete the job, if he's working alone, to the time it takes Worker II to complete the job, if he's working alone.
 - ב. The task that the two workers carry out together is the preparation of 300 replacement parts.
 Working together at their normal pace, the workers will need 6 days to complete this task.
 How many replacement parts can Worker I prepare in one day, if he's working alone at his normal pace?
2. The sequence a_n is given. The sum of the first n terms in the sequence is:
 $S_n = n^2 - 5n + [2 + 6 + 10 + \dots + (4n - 2)]$.
 - א. Find a formula for a_n , the general term of the given sequence.
 - ב. Study the terms of the given sequence whose individual values are less than 102 and calculate the largest value that can be produced by adding those terms (not necessarily the sum of all of those terms).

3. The organizers of the *Born to Sing* competition are in the process of deciding whether the competition will be judged only by Judge α or whether Judge α will be joined by two additional judges: Judge β and Judge γ . Judge α 's vote will be the same regardless of whether he judges alone or together with the other judges.
- Each judge's vote is independent of the votes of the other judges.
- If Judge α is the only judge, a contestant who receives his vote will advance to the next round of the competition.
- If there are three judges, a contestant will advance to the next round if at least 2 judges vote for him.
- Yossi is one of the contestants. We know that the probability that Judge α will vote for Yossi is equal to the probability that Judge β will vote for Yossi. The probability that Judge γ will vote for Yossi is 0.5.
- α . If the competition is judged only by Judge α , is the probability that Yossi will advance to the next round equal to the probability that Yossi will advance to the next round if the competition is judged by three judges? Explain.
- β . In the end, it was decided that the competition would be judged by three judges.
- Given: If we know that Yossi advanced to the next round, the probability that Judge α voted for him is greater than 0.8.
- Find the range of values of the probability that Judge α voted for Yossi.

Part Two — Planar Geometry and Planar Trigonometry

($33\frac{1}{3}$ points)

Answer two of the questions 4-6 (each question – $16\frac{2}{3}$ points).

Note! If you answer more than two questions, only the first two answers in your answer booklet will be graded.

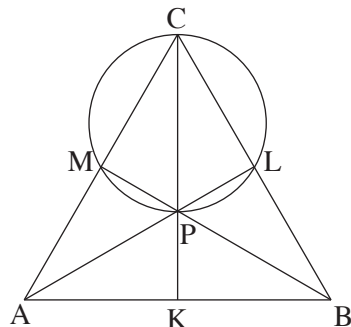
4. א. Prove that if a triangle has two equal medians [תיכונים], it is an isosceles triangle [משולש שווה-שוקיים].

- ב. In Triangle ABC, the points L, M and K are the midpoints of the sides CB, CA and AB, respectively.

Point P is the point at which the medians of the triangle intersect. Point P is located on a circle that passes through Points L, M and C (see figure).

Also given: $AL = BM$.

- (1) Prove that $BM \perp AC$.
- (2) Prove that $AK = AM$.



5. The quadrilateral AKLM is inscribed [חסום] in a circle.

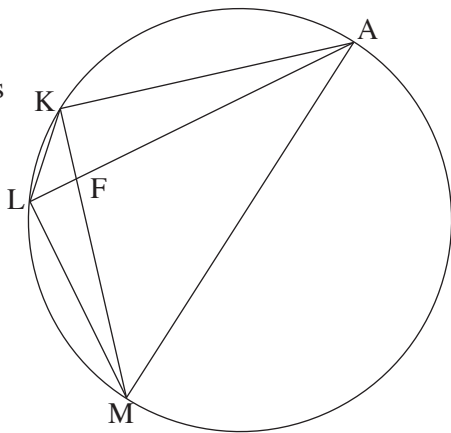
AM is a diameter of the circle.

The diagonals of the quadrilateral intersect at Point F (see figure).

Given: $ML = 30$ cm, $FL = a$ centimeters

The area of Triangle ALK is 3 times smaller than the area of Triangle ALM.

- א. Find the length of the height to Side LA in Triangle ALK.
- ב. Express the length of Segment KF in terms of a .
- ג. Prove that $\triangle AFM \sim \triangle KFL$.
- ד. Also given: $AF = 42.5$ cm; $ML > a$.
Find the value of a .



6. Point O is the center of the circle that is inscribed [חסום] inside Triangle ABC.

The extension of AO intersects
Side BC at Point E.

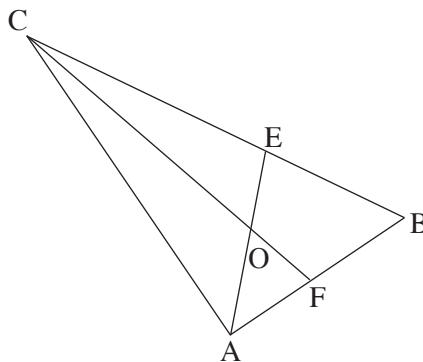
The extension of CO intersects
Side AB at Point F (see figure).

Given: $\angle BAC = \alpha$ and $\angle ABC = \beta$.

א. Express the ratio $\frac{AE}{CF}$ in terms of α and β .

ב. Also given: $\frac{AE}{CF} = \frac{1}{2}$; $\beta = 60^\circ$.

Show that a radius of the circle that circumscribes Triangle ACB equals $\frac{1}{2}BC$.



Part Three — Differential and Integral Calculus of Polynomials, Root Functions, Rational Functions and Trigonometric Functions (33 $\frac{1}{3}$ points)

Answer two of the questions 7-9 (each question – 16 $\frac{2}{3}$ points).

Note! If you answer more than two questions, only the first two answers in your answer booklet will be graded.

7. Given: The function $g(x) = \sin\left(\frac{2\pi}{3} - x\right)$ over the interval $0 \leq x \leq \frac{7}{3}\pi$.
- Find the points at which the graph of the function $g(x)$ intersects the axes.
 - Find the coordinates of the points at which the graph of the function $g(x)$ intersects the graph of the function $f(x) = \sin x$.
 - Point A is located on the graph of the function $g(x)$ and Point B is located on the graph of the function $f(x)$, such that the segment AB is parallel to the y-axis.
 - Find the maximum possible length of Segment AB.
 - How many segments like AB that have the maximum possible length are found in the given interval? Explain.

8. Two functions are given: $f(x) = x^2 + 4x + b$

$$g(x) = -x^2 + c$$

b and c are parameters greater than 0.

The graphs of the two functions have a common tangent at the shared point P.

- א. Express the coordinates of Point P, in terms of b if necessary.
- ב. On a single system of axes, sketch the graph of the function $f(x)$ and the graph of the function $g(x)$, if we know that $b > 4$.

The line $x = a$ intersects the common tangent at Point D, the graph of $f(x)$ at Point A and the graph of $g(x)$ at Point B.

(D, A and B are three different points.)

- ג. Show that the line PD is a median [תיכון] of Triangle PAB.
- ד. The area of the region bordered by the graph of $f(x)$, the common tangent and the lines $x = a$ and $x = -a$ equals S .

Express the size of the area bordered by the graph of $f(x)$, the graph of $g(x)$ and the lines $x = a$ and $x = -a$ in terms of S .

9. Given: The even function $f(x) = \sqrt{8 - ax + bx^2} + c$ is only defined over the interval $-2 \leq x \leq 2$.

a , b and c are parameters; $c > 0$.

א. Find the value of the parameter a and the value of parameter b .

Insert the value of a and the value of b , and then answer Items ב and ג.

ב. We draw a line that is a tangent to the graph of the function $f(x)$ at the point at which $x = \sqrt{2}$. We draw another line that is a tangent to the graph of the function at the point at which $x = -\sqrt{2}$.

The area of the region bordered by the two tangents and the x-axis is $\frac{49\sqrt{2}}{2}$.

Find the value of the parameter c .

ג. A function $g(x)$ is given over the interval $-2 \leq x \leq 2$, for which $g(x) = -f(x)$.

We draw a line that is a tangent to the graph of the function $g(x)$ at the point at which $x = \sqrt{2}$. We draw another line that is a tangent to the graph of the function at the point at which $x = -\sqrt{2}$.

What type of quadrilateral [מרובע] is formed by the two lines that are tangents to the graph of the function $f(x)$ and the two lines that are tangents to the graph of the function $g(x)$?

Explain.

Good Luck!

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בהצלחה!

זכות היוצרים שמורה למדינת ישראל
אין להעתיק או לפרסם אלא ברשות משרד החינוך